ISSN (e): 2250 – 3005 || Vol, 04 || Issue, 5 || May – 2014 ||
International Journal of Computational Engineering Research (IJCER)

Extensions of Enestrom-Kakeya Theorem

M. H. Gulzar
Department of Mathematics
University of Kashmir, Srinagar 190006

ABSTRACT:
In this paper we give an extension of the famous Enestrom-Kakeya Theorem, which generalizes many generalizations of the said theorem as well.

Mathematics Subject Classification: 30 C 10, 30 C 15

Keywords and Phrases: Coefficient, Polynomial, Zero

I. INTRODUCTION AND STATEMENT OF RESULTS

A famous result giving a bound for all the zeros of a polynomial with real positive monotonically decreasing coefficients is the following result known as Enestrom-Kakeya theorem [8]:

Theorem A: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that

$$a_n \geq a_{n-1} \geq ...... \geq a_1 \geq a_0 > 0 .$$

Then all the zeros of P(z) lie in the closed disk $|z| \leq 1$.

If the coefficients are monotonic but not positive, Joyal, Labelle and Rahman [6] gave the following generalization of Theorem A:

Theorem B: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that

$$a_n \geq a_{n-1} \geq ...... \geq a_1 \geq a_0 .$$

Then all the zeros of P(z) lie in the closed disk $|z| \leq \frac{a_n - a_0 + |a_0|}{|a_n|}$.

Aziz and Zargar [1] generalized Theorem B by proving the following result:

Theorem C: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some $k \geq 1$,

$$ka_n \geq a_{n-1} \geq ...... \geq a_1 \geq a_0 .$$

Then all the zeros of P(z) lie in the closed disk

$$|z+k-1| \leq \frac{ka_n - a_0 + |a_0|}{|a_n|} .$$

Gulz ar [4,5] generalized Theorem C to polynomials with complex coefficients and proved the following results:

Theorem D: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $\mathrm{Re}(a_j)=\alpha_j$, $\mathrm{Im}(a_j)=\beta_j, j=0,1,......, n$ such that for some $k \geq 1, 0 < \tau \leq 1$,

$$k\alpha_n \geq \alpha_{n-1} \geq ...... \geq \alpha_1 \geq \tau\alpha_0 .$$

Then all the zeros of P(z) lie in the closed disk

$$\left|z + (k-1)\frac{\alpha_n}{a_n}\right| \le \frac{k\alpha_n + 2|\alpha_0| - \tau(\alpha_0 + |\alpha_0|) + 2\sum_{j=0}^{n}|\beta_j|}{|a_n|}.$$

Theorem E: Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $\mathrm{Re}(a_j) = \alpha_j$,

$\mathrm{Im}(a_j) = \beta_j$, $j = 0,1,......, n$ such that for some $k \ge 1, 0 < \tau \le 1$,

$$k\beta_n \ge \beta_{n-1} \ge ...... \ge \beta_1 \ge \tau\beta_0.$$

Then all the zeros of P(z) lie in the closed disk

$$\left|z + (k-1)\frac{\beta_n}{a_n}\right| \le \frac{k\beta_n + 2|\beta_0| - \tau(\beta_0 + |\beta_0|) + 2\sum_{j=0}^{n}|\alpha_j|}{|a_n|}.$$

Theorem F: : Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some

real $\alpha, \beta; |\arg a_j - \beta| \le \alpha \le \frac{\pi}{2}$, $j = 0,1,2,......, n$, and for some $k \ge 1, 0 < \tau \le 1$,

$$k|a_n| \ge |a_{n-1}| \ge ...... \ge |a_1| \ge \tau|a_0|.$$

Then all the zeros of P(z) lie in the closed disk

$$|z| \le \frac{k|a_n|(1 + \cos + \sin\ \alpha) - |a_n| + 2|a_0| - \tau|\alpha_0|(\cos\ \alpha - \sin\ \alpha + 1) + 2\sin\ \alpha\sum_{j=1}^{n-1}|a_j|}{|a_n|}.$$

Some questions which have been raised by some researchers in connection with the Enestrom-Kakeya Theorem are[2]:

What happens , if (i) instead of the leading coefficient a_n, there is some a_j with

$a_{j+1} \ge a_j < a_{j-1}$ such that for some $k \ge 1$, $a_n \ge a_{n-1} \ge ...... \ge a_{j+1} \ge ka_j \ge a_{j-1} ...... \ge \alpha_1 \ge \alpha_0$,

j=1,2,…..,n and (ii) for some $k_1 \ge 1, k_2 \ge 1; k_1 a_n \ge k_2 a_{n-1} \ge ...... \ge a_1 \ge a_0$.

In this direction, Liman and Shah [7, Cor.1] have proved the following result:

Theorem G: Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some $k \ge 1$, ,

$$a_n \ge a_{n-1} \ge ...... \ge a_{\lambda+1} \ge ka_\lambda \ge a_{\lambda-1} ...... \ge a_1 \ge a_0.$$

Then P(z) has all its zeros in

$$|z| \le \frac{a_n - a_0 + |a_0| + (k-1)\left\{\sum_{j=\lambda}^{n}(a_j + |a_j|) - |a_n|\right\}}{|a_n|}.$$

Unfortunately, the conclusion of the theorem is not correct and their claim that it follows from Theorem 1 in [7] is false. The correct form of the result is as follows:

Theorem H: Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $\mathrm{Re}(a_j) = \alpha_j$,

$\mathrm{Im}(a_j) = \beta_j$, $j = 0,1,......, n$ such that for some $k \ge 1$, ,

$$a_n \ge a_{n-1} \ge ...... \ge a_{\lambda+1} \ge ka_\lambda \ge a_{\lambda-1} ...... \ge a_1 \ge a_0.$$

Then P(z) has all its zeros in

$$|z| \le \frac{a_n - a_0 + |a_0| + 2(k-1)|a_\lambda|}{|a_n|}.$$

In this paper , we are going to prove the following more general result:

Theorem 1: Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some $k \ge 1, 0 < \tau \le 1$,

$$a_n \ge a_{n-1} \ge ...... \ge a_{\lambda+1} \ge ka_\lambda \ge a_{\lambda-1} ...... \ge a_1 \ge \tau a_0 .$$

Then P(z) has all its zeros in

$$|z| \le \frac{a_n + 2(k-1)|a_\lambda| - \tau(a_0 + |a_0|) + 2|a_0|}{|a_n|} .$$

Remark 1: For $\tau = 1$,Theorem 1 reduces to Theorem H.

Taking in particular $k = \frac{a_{\lambda-1}}{a_\lambda} \ge 1$ in Theorem 1, we get the following

Corollary 1: Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $\text{Re}(a_j) = \alpha_j$,

$\text{Im}(a_j) = \beta_j$, $j = 0,1,......, n$ such that

$$a_n \ge a_{n-1} \ge ...... \ge a_{\lambda+1} \ge a_\lambda \le a_{\lambda-1}. \ge ..... \ge a_1 \ge \tau a_0 .$$

Then P(z) has all its zeros in

$$|z| \le \frac{a_n + 2(\frac{a_{\lambda-1} - a_\lambda}{a_\lambda})|a_\lambda| - \tau(a_0 + |a_0|) + 2|a_0|}{|a_n|} .$$

For $\tau = 1$, Cor. 1 reduces to the following

Corollary 2: Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $\text{Re}(a_j) = \alpha_j$,

$\text{Im}(a_j) = \beta_j$, $j = 0,1,......, n$ such that

$$a_n \ge a_{n-1} \ge ...... \ge a_{\lambda+1} \ge a_\lambda \le a_{j-1}. \ge ..... \ge a_1 \ge a_0 .$$

Then P(z) has all its zeros in

$$|z| \le \frac{a_n + (\frac{a_{\lambda-1} - a_\lambda}{a_\lambda})|a_\lambda| - a_0 + |a_0|)}{|a_n|} .$$

Theorem 1 is a special case of the following more general result:

Theorem 2: Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $\text{Re}(a_j) = \alpha_j$,

$\text{Im}(a_j) = \beta_j$, $j = 0,1,......, n$ such that for some $k \ge 1$,

$$\alpha_n \ge \alpha_{n-1} \ge ...... \ge \alpha_{\lambda+1} \ge k\alpha_\lambda \ge \alpha_{\lambda-1} ...... \ge \alpha_1 \ge \tau\alpha_0 .$$

Then P(z) has all its zeros in

$$|z| \le \frac{\alpha_n + 2(k-1)|\alpha_\lambda| - \tau(\alpha_0 + |\alpha_0|) + 2|\alpha_0| + 2\sum_{j=0}^{n}|\beta_j|}{|a_n|} .$$

Remark 2: If $\beta_j = 0, \forall j = 0,1,2,......, n$ i.e. a_j is real, then Theorem 2 reduces to Theorem 1.

Applying Theorem 2 to the polynomial –iP(z), we get the following result:

Theorem 3: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $\mathrm{Re}(a_j)=\alpha_j$, $\mathrm{Im}(a_j)=\beta_j$, $j=0,1,......,n$ such that for some $k\geq 1, 0<\tau\leq 1$,

$$\beta_n \geq \beta_{n-1} \geq ...... \geq \beta_{j+1} \geq k\beta_j \geq \beta_{j-1} ...... \geq \beta_1 \geq \tau\beta_0 .$$

Then P(z) has all its zeros in

$$|z| \leq \frac{a_n + 2(k-1)|\beta_\lambda| - \tau(\beta_0 + |\beta_0|) + 2|\beta_0| + 2\sum_{=0\, j}^{n} |\alpha_j|}{|a_n|} .$$

For polynomials with complex coefficients, we have the following form of Theorem 1:

Theorem 4: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some $k\geq 1, 0<\tau\leq 1$,

$$|a_n| \geq |a_{n-1}| \geq ...... \geq |a_{\lambda+1}| \geq k|a_\lambda| \geq |a_{\lambda-1}| ...... \geq |a_1| \geq \tau|a_0| .$$

Then P(z) has all its zeros in

$$|z| \leq \frac{1}{|a_n|}[|a_n|(\cos\alpha + \sin\alpha) - k|a_\lambda|(\cos\alpha - \sin\alpha - 1) + 2|a_\lambda|(k + k\sin\alpha - 1)$$

$$- \tau|a_0|(\cos\alpha - \sin\alpha + 1) + 2|a_0|]$$

Remark 3: For k=1, Theorem 4 reduces to Theorem F with k=1.
Next, we prove the following result:

Theorem 5: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $\mathrm{Re}(a_j)=\alpha_j$, $\mathrm{Im}(a_j)=\beta_j$, $j=0,1,......,n$ such that for some $k_1\geq 1, k_2\geq 1, 0<\tau\leq 1$,

$$k_1\alpha_n \geq k_2\alpha_{n-1} \geq \alpha_{n-2} ...... \alpha_1 \geq \tau\alpha_0 .$$

Then P(z) has all its zeros in

$$|z| \leq \frac{(k_1|\alpha_n| + k_2|\alpha_{n-1}|) + (k_1\alpha_n - k_2\alpha_{n-1}) + \alpha_{n-1} - (|\alpha_n| + |\alpha_{n-1}|) + 2|\alpha_0| + 2\sum_{j=0}^{n} |\beta_j|}{|a_n|} .$$

Remark 4: If $\beta_j = 0, \forall j = 0,1,2,......,n$ i.e. a_j is real, we get the following result:

Corollary 3: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some $k_1\geq 1, k_2\geq 1, 0<\tau\leq 1$,

$$k_1 a_n \geq k_2 a_{n-1} \geq a_{n-2} ...... a_1 \geq \tau a_0 .$$

Then P(z) has all its zeros in

$$|z| \leq \frac{(k_1|a_n| + k_2|a_{n-1}|) + (k_1 a_n - k_2 a_{n-1}) + a_{n-1} - (|a_n| + |a_{n-1}|) + 2|a_0|}{|a_n|} .$$

Applying Theorem 2 to the polynomial –iP(z), we get the following result from Theorem 4:

Theorem 6: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $\mathrm{Re}(a_j)=\alpha_j$, $\mathrm{Im}(a_j)=\beta_j$, $j=0,1,......,n$ such that for some $k_1\geq 1, k_2\geq 1, 0<\tau\leq 1$,

$$k_1\beta_n \geq k_2\beta_{n-1} \geq \beta_{n-2}...... \geq \beta_1 \geq \tau\beta_0 .$$

Then P(z) has all its zeros in

$$|z| \leq \frac{(k_1|\beta_n| + k_2|\beta_{n-1}|) + (k_1\beta_n - k_2\beta_{n-1}) + \beta_{n-1} - (|\beta_n| + |\beta_{n-1}|) + 2|\beta_0| + 2\sum_{j=0}^{n}|\alpha_j|}{|a_n|} .$$

Theorem 7: Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some

real $\alpha, \beta; \left|\arg a_j - \beta\right| \leq \alpha \leq \frac{\pi}{2}, j = 0,1,2,......, n$, and for some $k_1 \geq 1, k_2 \geq 1, 0 < \tau \leq 1$,

$$k_1|a_n| \geq k_2|a_{n-1}| \geq |a_{n-2}|...... \geq |a_1| \geq \tau|a_0| .$$

Then P(z) has all its zeros in

$$|z| \leq \frac{1}{|a_n|}\begin{bmatrix} k_1|a_n|(1+\cos\alpha+\sin\alpha) + k_2|a_{n-1}|(1-\cos\alpha+\sin\alpha) - |a_n| - |a_{n-1}|(1-\cos\alpha) + 2|a_0| \\ -\tau|a_0|(\cos\alpha - \sin\alpha + 1) + 2\sin\alpha\sum_{j=1}^{n-2}|a_j| \end{bmatrix} .$$

Remark 4: For $k_1 = k, k_2 = 1$, Theorem 6 reduces to Theorem F.

Taking $\tau = 1$ in Theorem 7, we get the following

Corollary 4: Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some

real $\alpha, \beta; \left|\arg a_j - \beta\right| \leq \alpha \leq \frac{\pi}{2}, j = 0,1,2,......, n$, and for some $k_1 \geq 1, k_2 \geq 1,$,

$$k_1|a_n| \geq k_2|a_{n-1}| \geq |a_{n-2}|...... \geq |a_1| \geq |a_0| .$$

Then P(z) has all its zeros in

$$|z| \leq \frac{1}{|a_n|}[k_1|a_n|(1+\cos\alpha+\sin\alpha) + k_2|a_{n-1}|(1-\cos\alpha+\sin\alpha) - |a_n|$$

$$-|a_{n-1}|(1-\cos\alpha) - \tau|a_0|(1+\cos\alpha-\sin\alpha) + 2|a_0| + 2\sin\alpha\sum_{j=1}^{n-1}|a_j|] .$$

II. LEMMA

For the proof of Theorem 6, we need the following lemma:

Lemma: Let a_1 and a_2 be any two complex numbers such that $|a_1| \geq |a_2|$ and for some real numbers α and β , $\left|\arg a_j - \beta\right| \leq \alpha \leq \frac{\pi}{2}, j = 1,2$,then

$$|a_1 - a_2| \leq (|a_1| - |a_2|)\cos\alpha + (|a_1| + |a_2|)\sin\alpha .$$

The above lemma is due to Govil and Rahman [3] .

3. Proofs of Theorems

Proof of Theorem 2: Consider the polynomial

$$F(z) = (1-z)P(z) = (1-z)(a_n z^n + a_{n-1}z^{n-1} + ... + a_1 z + a_0)$$

$$= -a_n z^{n+1} + (a_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + ... + (a_{\lambda+1} - a_\lambda)z^{\lambda+1}$$

$$+ (a_\lambda - a_{\lambda-1})z^\lambda + ... + (a_1 - a_0)z + a_0$$

$$= -a_n z^{n+1} + (\alpha_n - \alpha_{n-1}) z^n + (\alpha_{n-1} - \alpha_{n-2}) z^{n-1} + \ldots\ldots$$

$$+ \{(\alpha_{\lambda+1} - k\alpha_\lambda) + (k\alpha_\lambda - \alpha_\lambda)\} z^{\lambda+1} + \{(k\alpha_\lambda - \alpha_{\lambda-1}) - (k\alpha_\lambda - \alpha_\lambda)\} z^\lambda + \ldots\ldots$$

$$+ \{(\alpha_1 - \tau\alpha_0) + (\tau\alpha_0 - \alpha_0)\} z + \alpha_0 + i\{(\beta_n - \beta_{n-1}) z^n + \ldots\ldots(\beta_1 - \beta_0) z + \beta_0\}$$

For $|z| > 1$, we have, $\frac{1}{|z|^j} < 1, \forall j = 1,2,\ldots\ldots, \ n$ so that, by using the hypothesis,

$$|F(z)| \geq |a_n||z|^{n+1} - [|\alpha_n - \alpha_{n-1}||z|^n + |\alpha_{n-1} - \alpha_{n-1}||z|^{n-1} + \ldots\ldots + |\alpha_{\lambda+1} - k\alpha_\lambda||z|^{\lambda+1}$$

$$+ (k-1)|\alpha_\lambda||z|^{\lambda+1} + |k\alpha_\lambda - \alpha_{\lambda-1}||z|^\lambda + (k-1)|\alpha_\lambda||z|^\lambda + \ldots\ldots + |\alpha_1 - \tau a_0||z|$$

$$+ (1-\tau)|\alpha_0||z| + |\alpha_0| + \sum_{j=1}^{n} |\beta_j - \beta_{j-1}||z|^j + |\beta_0|$$

$$= |z|^n [|a_n||z| - \{[|\alpha_n - \alpha_{n-1}| + |\alpha_{n-1} - \alpha_{n-1}| \frac{1}{|z|} + \ldots\ldots + |\alpha_{\lambda+1} - k\alpha_\lambda| \frac{1}{|z|^{n-\lambda-1}}$$

$$+ (k-1)|\alpha_\lambda| \frac{1}{|z|^{n-\lambda-`1}} + |k\alpha_\lambda - \alpha_{\lambda-1}| \frac{1}{|z|^{n-\lambda}} + (k-1)|\alpha_\lambda| \frac{1}{|z|^{n-\lambda}} + \ldots\ldots + |\alpha_1 - \tau a_0| \frac{1}{|z|^{n-1}}$$

$$+ (1-\tau)|\alpha_0| \frac{1}{|z|^{n-1}} + |\alpha_0| \frac{1}{|z|^n} + \sum_{j=1}^{n} |\beta_j - \beta_{j-1}| \frac{1}{|z|^{n-j}} + |\beta_0| \frac{1}{|z|^n}\}]$$

$$> |z|^n [|a_n||z| - [\alpha_n - \alpha_{n-1} + \alpha_{n-1} - \alpha_{n-2} + \ldots\ldots + \alpha_{\lambda+1} - k\alpha_\lambda + (k-1)|\alpha_\lambda|$$

$$+ k\alpha_\lambda - \alpha_{\lambda-1} + (k-1)|\alpha_\lambda| + \ldots\ldots + \alpha_1 - \tau\alpha_0 + (1-\tau)|\alpha_0| + |\alpha_0|$$

$$+ \sum_{j=1}^{n} (|\beta_j| + |\beta_{j-1}|) + |\beta_0|]$$

$$= |z|^n [|a_n||z| - [\alpha_n + 2(k-1)|\alpha_\lambda| - \tau(\alpha_0 + |\alpha_0|) + 2|\alpha_0| + 2\sum_{j=0}^{n} |\beta_j|]$$

$$> 0$$

if

$$|z| > \frac{\alpha_n + 2(k-1)|\alpha_\lambda| - \tau(\alpha_0 + |\alpha_0|) + 2|\alpha_0| + 2\sum_{j=0}^{n} |\beta_j|}{|a_n|}.$$

This shows that the zeros of F(z) having modulus greater than 1 lie in

$$|z| \leq \frac{\alpha_n + 2(k-1)|\alpha_\lambda| - \tau(\alpha_0 + |\alpha_0|) + 2|\alpha_0| + 2\sum_{j=0}^{n} |\beta_j|}{|a_n|}.$$

But the zeros of F(z) whose modulus is less than or equal to 1 already satisfy the above inequality. Hence, it follows that all the zeros of F(z) lie in

$$|z| \leq \frac{\alpha_n + 2(k-1)|\alpha_\lambda| - \tau(\alpha_0 + |\alpha_0|) + 2|\alpha_0| + 2\sum_{j=0}^{n} |\beta_j|}{|a_n|}.$$

Since the zeros of P(z) are also the zeros of F(z), the result follows.

Proof of Theorem 4: : Consider the polynomial

$$F(z) = (1-z)P(z) = (1-z)(a_n z^n + a_{n-1}z^{n-1} + ... + a_1 z + a_0)$$

$$= -a_n z^{n+1} + (a_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + ... + \{(a_{\lambda+1} - ka_\lambda) + (ka_\lambda - a_\lambda)\} z^{\lambda+1}$$

$$+ \{(ka_\lambda - a_{\lambda-1}) - (ka_\lambda - a_\lambda)\} z^\lambda + (a_{\lambda-1} - a_{\lambda-2}) z^{\lambda-1} + ...$$

$$\{(a_1 - \tau a_0) + (\tau a_0 - a_0)\} z + a_0 .$$

For $|z| > 1$, we have, $\frac{1}{|z|^j} < 1, \forall j = 1,2,......., \ n$, so that, by using the hypothesis and the Lemma,

$$|F(z)| \geq |a_n||z|^{n+1} - [|a_n - a_{n-1}||z|^n + |a_{n-1} - a_{n-1}||z|^{n-1} + ... + |a_{\lambda+1} - ka_\lambda||z|^{\lambda+1}$$

$$+ (k-1)|a_\lambda||z|^{\lambda+1} + |ka_\lambda - a_{\lambda-1}||z|^\lambda + (k-1)|a_\lambda||z|^\lambda + ... + |a_1 - \tau a_0||z|$$

$$+ (1-\tau)|a_0||z| + |a_0|$$

$$= |z|^n [|a_n||z| - \{|a_n - a_{n-1}| + |a_{n-1} - a_{n-2}|\frac{1}{|z|} + ... + |a_{\lambda+1} - ka_\lambda|\frac{1}{|z|^{n-\lambda-1}}$$

$$+ (k-1)|a_\lambda|\frac{1}{|z|^{n-\lambda-1}} + |ka_\lambda - a_{\lambda-1}|\frac{1}{|z|^{n-\lambda}} + (k-1)|a_\lambda|\frac{1}{|z|^{n-\lambda}} + ... + |a_1 - \tau a_0|\frac{1}{|z|^{n-1}}$$

$$+ (1-\tau)|a_0|\frac{1}{|z|^{n-1}} + |a_0|\frac{1}{|z|^n}\}]$$

$$> |z|^n [|a_n||z| - \{|a_n - a_{n-1}| + |a_{n-1} - a_{n-2}| + ... + |a_{\lambda+1} - ka_\lambda|$$

$$+ (k-1)|a_\lambda| + |ka_\lambda - a_{\lambda-1}| + (k-1)|a_\lambda| + ... + |a_1 - \tau a_0|$$

$$+ (1-\tau)|a_0| + |a_0|\}]$$

$$\geq |z|^n [|a_n||z| - \{(|a_n| - |a_{n-1}|)\cos\alpha + (|a_n| + |a_{n-1}|)\sin\alpha + (|a_{n-1}| - |a_{n-2}|)\cos\alpha$$

$$+ (|a_{n-1}| + |a_{n-2}|)\sin\alpha + ... + (|a_{\lambda+1}| - k|a_\lambda|)\cos\alpha + (|a_{\lambda+1}| + k|a_\lambda|)\sin\alpha$$

$$+ (k-1)|a_\lambda| + (k|a_\lambda| - |a_{\lambda+1}|)\cos\alpha + (k|a_\lambda| + |a_{\lambda+1}|)\sin\alpha + (k-1)|a_\lambda|$$

$$+ (|a_{\lambda+1}| - |a_{\lambda-2}|)\cos\alpha + (|a_{\lambda+1}| + |a_{\lambda-2}|)\sin\alpha + ...(|a_1| - \tau|a_0|)\cos\alpha$$

$$+ (|a_1| + \tau|a_0|)\sin\alpha + (1-\tau)|a_0| + |a_0|\}]$$

$$= |z|^n [|a_n||z| - \{|a_n|(\cos\alpha + \sin\alpha) - k|a_\lambda|(\cos\alpha - \sin\alpha - 1) + 2|a_\lambda|(k + k\sin\alpha - 1)$$

$$- \tau|a_0|(\cos\alpha - \sin\alpha + 1) + 2|a_0|\}]$$

$$> 0$$

if

$$|z| > \frac{1}{|a_n|}[|a_n|(\cos\alpha + \sin\alpha) - k|a_\lambda|(\cos\alpha - \sin\alpha - 1) + 2|a_\lambda|(k + k\sin\alpha - 1)$$

$$- \tau|a_0|(\cos\alpha - \sin\alpha + 1) + 2|a_0|]$$

This shows that the zeros of F(z) having modulus greater than 1 lie in

$$|z| \le \frac{1}{|a_n|}[|a_n|(\cos\alpha + \sin\alpha) - k|a_\lambda|(\cos\alpha - \sin\alpha - 1) + 2|a_\lambda|(k + k\sin\alpha - 1)$$

$$- \tau|a_0|(\cos\alpha - \sin\alpha + 1) + 2|a_0|]$$

But the zeros of F(z) whose modulus is less than or equal to 1 already satisfy the above inequality. Hence, it follows that all the zeros of F(z) lie in

$$|z| \le \frac{1}{|a_n|}[|a_n|(\cos\alpha + \sin\alpha) - k|a_\lambda|(\cos\alpha - \sin\alpha - 1) + 2|a_\lambda|(k + k\sin\alpha - 1)$$

$$- \tau|a_0|(\cos\alpha - \sin\alpha + 1) + 2|a_0|]$$

Since the zeros of P(z) are also the zeros of F(z), the result follows.

Proof of Theorem 5: Consider the polynomial

$$F(z) = (1 - z)P(z) = (1 - z)(a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0)$$

$$= -a_n z^{n+1} + (a_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + ... + (a_1 - a_0)z + a_0$$

$$= -a_n z^{n+1} + \{(k_1\alpha_n - k_2\alpha_{n-1}) - (k_1\alpha_n - \alpha_n) + (k_2\alpha_{n-1} - \alpha_{n-1})\} z^n + (\alpha_{n-1} - \alpha_{n-2})z^{n-1}$$

$$+ ...... + \{(\alpha_1 - \tau\alpha_0) + (\tau\alpha_0 - \alpha_0)\} z + \alpha_0 + i\{(\beta_n - \beta_{n-1})z^n + ...$$

$$+ (\beta_1 - \beta_0)z + \beta_0\}$$

For $|z| > 1$, we have, $\frac{1}{|z|^j} < 1, \forall j = 1,2,......., n$, so that, by using the hypothesis,

$$|F(z)| \ge |a_n||z|^{n+1} - [|k_1\alpha_n - k_2\alpha_{n-1}||z|^n + |k_1\alpha_n - \alpha_n||z|^n + |k_2\alpha_{n-1} - \alpha_{n-1}||z|^n + |\alpha_{n-1} - \alpha_{n-2}||z|^{n-1}$$

$$+ ...... + |\alpha_1 - \tau a_0||z| + (1 - \tau)|\alpha_0||z| + |\alpha_0| + \sum_{j=1}^{n} |\beta_j - \beta_{j-1}||z|^j + |\beta_0|]$$

$$= |z|^n [|a_n||z| - [|k_1\alpha_n - k_2\alpha_{n-1}| + |k_1\alpha_n - \alpha_n| + |k_2\alpha_{n-1} - \alpha_{n-1}| + |\alpha_{n-1} - \alpha_{n-2}|\frac{1}{|z|} ......$$

$$+ |\alpha_1 - \tau\alpha_0|\frac{1}{|z|^{n-1}} + (1 - \tau)|\alpha_0|\frac{1}{|z|^{n-1}} + |\alpha_0|\frac{1}{|z|^n} + \sum_{j=1}^{n} |\beta_j - \beta_{j-1}|\frac{1}{|z|^{n-j}} + |\beta_0|\frac{1}{|z|^n}\}]$$

$$> |z|^n [|a_n||z| - \{k_1\alpha_n - k_2\alpha_{n-1} + (k_1 - 1)|\alpha_n| + (k_2 - 1)|\alpha_{n-1}| + \alpha_{n-1} - \alpha_{n-2} + ...$$

$$+ \alpha_1 - \tau\alpha_0 + (1 - \tau)|\alpha_0| + |\alpha_0| + \sum_{j=1}^{n} (|\beta_j| + |\beta_{j-1}|) + |\beta_0|\}]$$

$$= |z|^n [|a_n||z| - \{k_1\alpha_n - k_2\alpha_{n-1} + (k_1 - 1)|\alpha_n| + (k_2 - 1)|\alpha_{n-1}| + \alpha_{n-1} + ...$$

$$- \tau(\alpha_0 + |\alpha_0|) + 2|\alpha_0| + 2\sum_{j=0}^{n} |\beta_j|\}]$$

$$= |z|^n [|a_n||z| - \{(k_1|\alpha_n| + k_2|\alpha_{n-1}|) + (k_1\alpha_n - k_2\alpha_{n-1}) + \alpha_{n-1} - (|\alpha_n| + |\alpha_{n-1}|)$$

$$+ 2|\alpha_0| + 2\sum_{j=0}^{n} |\beta_j|\}]$$

$$> 0$$

if

$$|z| > \frac{1}{|a_n|}[(k_1|\alpha_n| + k_2|\alpha_{n-1}|) + (k_1\alpha_n - k_2\alpha_{n-1}) + \alpha_{n-1} - (|\alpha_n| + |\alpha_{n-1}|) + 2|\alpha_0| + 2\sum_{j=0}^{n}|\beta_j|]$$

This shows that the zeros of F(z) having modulus greater than 1 lie in

$$|z| \le \frac{1}{|a_n|}[(k_1|\alpha_n| + k_2|\alpha_{n-1}|) + (k_1\alpha_n - k_2\alpha_{n-1}) + \alpha_{n-1} - (|\alpha_n| + |\alpha_{n-1}|) + 2|\alpha_0| + 2\sum_{j=0}^{n}|\beta_j|] .$$

But the zeros of F(z) whose modulus is less than or equal to 1 already satisfy the above inequality. Hence, it follows that all the zeros of F(z) lie in

$$|z| \le \frac{1}{|a_n|}[(k_1|\alpha_n| + k_2|\alpha_{n-1}|) + (k_1\alpha_n - k_2\alpha_{n-1}) + \alpha_{n-1} - (|\alpha_n| + |\alpha_{n-1}|) + 2|\alpha_0| + 2\sum_{j=0}^{n}|\beta_j|]$$

Since the zeros of P(z) are also the zeros of F(z), Theorem 4 follows.

Proof of Theorem 6. Consider the polynomial

$$F(z) = (1-z)P(z) = (1-z)(a_n z^n + a_{n-1}z^{n-1} + \ldots\ldots + a_1 z + a_0)$$
$$= -a_n z^{n+1} + (a_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + \ldots\ldots + (a_1 - a_0)z + a_0$$
$$= -a_n z^{n+1} + \{(k_1 a_n - k_2 a_{n-1}) - (k_1 a_n - a_n) + (k_2 a_{n-1} - a_{n-1})\} z^n + (a_{n-1} - a_{n-2})z^{n-1}$$
$$+ \ldots\ldots + \{(a_1 - \tau a_0) + (\tau a_0 - a_0)\} z + a_0$$

For $|z| > 1$, we have, $\frac{1}{|z|^j} < 1, \forall j = 1,2,\ldots\ldots, n$, so that, by using the hypothesis and the Lemma,

$$|F(z)| \ge |a_n||z|^{n+1} - [|k_1 a_n - k_2 a_{n-1}||z|^n + |k_1 a_n - \alpha_n||z|^n + |k_2 a_{n-1} - a_{n-1}||z|^n + |a_{n-1} - a_{n-2}||z|^{n-1}$$
$$+ \ldots\ldots + |a_1 - \tau a_0||z| + (1-\tau)|a_0||z| + |a_0|]$$

$$= |z|^n [|a_n||z| - [|k_1 a_n - k_2 a_{n-1}| + |k_1 a_n - a_n| + |k_2 a_{n-1} - a_{n-1}| + |a_{n-1} - a_{n-2}|\frac{1}{|z|} + \ldots\ldots$$
$$+ |a_1 - \tau a_0|\frac{1}{|z|^{n-1}} + (1-\tau)|a_0|\frac{1}{|z|^{n-1}} + |a_0|\frac{1}{|z|^n}]$$
$$> |z|^n [|a_n||z| - \{|k_1 a_n - k_2 a_{n-1}| + |k_1 a_n - a_n| + |k_2 a_{n-1} - a_{n-1}| + |a_{n-1} - a_{n-2}| + \ldots\ldots$$
$$+ |a_1 - \tau a_0| + (1-\tau)|a_0| + |a_0|\}]$$
$$\ge |z|^n [|a_n||z| - \{(k_1|a_n| - k_2|a_{n-1}|)\cos\alpha + (k_1|a_n| + k_2|a_{n-1}|)\sin\alpha + (k_1 - 1)|a_n|$$
$$+ (k_2 - 1)|a_{n-1}| + (|a_{n-1}| - |a_{n-2}|)\cos\alpha + (|a_{n-1}| + |a_{n-2}|)\sin\alpha + \ldots\ldots$$
$$+ (|a_1| - \tau|a_0|)\cos\alpha + (|a_1| + \tau|a_0|)\sin\alpha + (1-\tau)|a_0|\}]$$

$$= |z|^n [|a_n||z| - \{k_1|a_n|(1 + \cos\alpha + \sin\alpha) + k_2|a_{n-1}|(1 - \cos\alpha + \sin\alpha) - |a_n|$$
$$- |a_{n-1}|(1 - \cos\alpha) - \tau|a_0|(1 + \cos\alpha - \sin\alpha) + 2|a_0| + 2\sin\alpha\sum_{j=1}^{n-1}|a_j|\}]$$

$$>0$$

if

$$|z| > \frac{1}{|a_n|}[k_1|a_n|(1+\cos\alpha+\sin\alpha)+k_2|a_{n-1}|(1-\cos\alpha+\sin\alpha)-|a_n|$$

$$-|a_{n-1}|(1-\cos\alpha)-\tau|a_0|(1+\cos\alpha-\sin\alpha)+2|a_0|+2\sin\alpha\sum_{j=1}^{n-1}|a_j|].$$

This shows that the zeros of F(z) having modulus greater than 1 lie in

$$|z| \leq \frac{1}{|a_n|}[k_1|a_n|(1+\cos\alpha+\sin\alpha)+k_2|a_{n-1}|(1-\cos\alpha+\sin\alpha)-|a_n|$$

$$-|a_{n-1}|(1-\cos\alpha)-\tau|a_0|(1+\cos\alpha-\sin\alpha)+2|a_0|+2\sin\alpha\sum_{j=1}^{n-1}|a_j|].$$

But the zeros of F(z) whose modulus is less than or equal to 1 already satisfy the above inequality. Hence, it follows that all the zeros of F(z) lie in

$$|z| \leq \frac{1}{|a_n|}[k_1|a_n|(1+\cos\alpha+\sin\alpha)+k_2|a_{n-1}|(1-\cos\alpha+\sin\alpha)-|a_n|$$

$$-|a_{n-1}|(1-\cos\alpha)-\tau|a_0|(1+\cos\alpha-\sin\alpha)+2|a_0|+2\sin\alpha\sum_{j=1}^{n-1}|a_j|].$$

Since the zeros of P(z) are also the zeros of F(z), the result follows.

REFERENCES

[1] A.Aziz and B.A.Zargar, Some Extensions of Enestrom-Kakeya Theorem, Glasnik Math. 31(1996), 239-244.
[2] N. K. Govvil, International Congress of ASIAM, Jammu University, India March 31-April 3, 2007.
[3] N. K. Govil and Q.I. Rahman, On the Enestrom-Kakeya Theorem, Tohoku Math.J.20 (1968), 126-136.
[4] M. H. Gulzar, Some Refinements of Enestrom-Kakeya Theorem, Int. Journal of Mathematical Archive -2(9, 2011),1512-1529.
[5] M. H. Gulzar, Ph.D thesis, Department of Mathematics, University of Kashmir Srinagar, 2012.
[6] A. Joyal, G. Labelle and Q. I. Rahman, On the Location of Zeros of Polynomials, Canad. Math. Bull., 10(1967), 53-66.
[7] A. Liman and W. M. Shah, Extensions of Enestrom-Kakeya Theorem, Int. Journal of Modern Mathematical Sciences, 2013, 8(2), 82-89.
[8] M. Marden, Geometry of Polynomials, Math. Surveys, No.3; Amer. Math. Soc. Providence R.I. 1966.