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Interpolation sets in spaces of continuous metric-valuedfunctions

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ABSTRACT
Let Xand Kbe a Čech-complete topological group and a compact group, respectively. We prove that if Gis a non-equicontinuous subset of CHom(X, K), the set of all continuous homomorphisms of Xinto K, then there is a countably infinite subset L ⊆Gsuch that LKXis canonically homeomorphic to βω, the Stone–Čech compactifcation of the natural numbers. As a consequence, if Gis an infinite subset of CHom(X, K)such that for every countable subset L ⊆Gand compact separable subset Y⊆Xit holds that either LKYhas countable tightness or |LKY| ≤c, then Gis equicontinuous. Given a topological group G, denote by G+the (algebraic) group Gequipped with the Bohr topology. It is said that Grespectsa topological property Pwhen Gand G+have the same subsets satisfying P. As an application of our main result, we prove that if Gis an abelian, locally quasiconvex, locally kωgroup, then the following holds: (i) Grespects any compact-like property Pstronger than or equal to functional boundedness; (ii) Gstrongly respects compactness.

Keywords: *Čech-complete group Locally kω-group Interpolation set Bohr compactification Bohr topology Respectscompactness*

I. INTRODUCTION

Let Gbe a locally compact abelian group and let _Gdenote its Pontryagin dual group. We say that a subset Eof _Gis *Sidon*if for every bounded function fon Ethere corresponds a Borel regular measure on G, μ, such that _μ(γ) =f(γ)for all γ∈E(here _μdenotes the Fourier transform of μ). If, in addition, μis assumed to be discrete (it has a countable support) then it is said that Eis an I_0-*set*. Therefore, each I_0-set is Sidon. For instance, lacunary (or Hadamard) sets (i.e. sequences $(z_n)_{n}$⊆Nsuch that infz_{n+1}/z_n>1) are perhaps the simplest examples of I_0-sets. The search for interpolation sets is a main goal in harmonic analysis and the monograph by Graham and Hare [25]contains most of the recent results in this area.

In this paper, this question is approached from a topological viewpoint that is based on the equivalent formulation of this notion given by Hartman and Ryll-Nardzewski [26](in fact, the term I_0-set is due to them). According to their (equivalent) definition a subset Eof a locally compact abelian group Gis an I_0-set if for each f∈l_∞(E)(that is, for each complex-valued, bounded function defined on E) there exists an almost periodic function $f_b$on Gsuch that f(γ) =f_b(γ)for all γ∈E. Furthermore, since every almost periodic function on a topological group Gis the restriction of a continuous function defined on the Bohr compactification bGof G, it follows that E⊆Gis an I_0-set if each f∈l_∞(E)can be extended to a continuous function $f_b$defined on bG. The latter property implies that $E^{bG}$is canonically homeomorphic to βE, the Stone–Čech compactification of the set Eequipped with the discrete topology. This equivalent definition of I_0-set and the *duality methods*obtained from Pontryagin–van Kampen duality allows us to apply topological techniques in the investigation of this family of interpolation sets. Thus, we can prove the existence of I_0-sets for much larger classes of groups than locally compact abelian groups. Several applications of our results to different questions related to the Bohr compactification and topology of topological abelian groups are also obtained. Last but not least, we deal with the topological properties of sets of continuous functions. Indeed, if Xand Mare a topological space and a metrizable space respectively, given a subset G ⊆C(X, M), we look at the possible existence of copies of βω(the Stone–Čech compactification of the natural numbers) within G^{M^X}. This property, or its absence, has deep implications on the topological structure of Gas a set of continuous functions on X and has found many applications in diff erent settings (for instance, see [20,22,17,25] where there are applications to topological groups, dynamical systems, functional analysis and harmonic analysis, respectively).

The starting point of this paper stems from a celebrated theorem by Bourgain, Fremlin and Talagrand about compact subsets of Baire class 1 functions [7], that we present in the way it is formulated by Todorčević in [35].

Theorem 1.1. (J. Bourgain, D.H. Fremlin, M. Talagrand) *Let X be a Polish space and let $\{f_n\}_{n<\omega} \subseteq C(X)$ be a pointwise bounded sequence. The following assertions are equivalent (where the closure is taken in $\mathbb{R}^X$):*

(a) *$\{f_n\}_{n<\omega}$ is sequentially dense in its closure.*
(b) *The closure of $\{f_n\}_{n<\omega}$ contains no copy of $\beta\omega$.*

A variant of this result is due to Pol [33, p. 34], that again was formulated in diff erent terms (cf. [9]). Here $B_1(X)$ denotes the set of all Baire class 1 functions defined on X.

Theorem 1.2. (R. Pol) *Let X be a complete metric space, G a subset of $C(X)$, which is uniformly bounded, and $K = \overline{G}$ the closure of G in $\mathbb{R}^X$. Then the following are equivalent:*

(a) *$K \not\subset B_1(X)$.*
(b) *G contains a sequence whose closure in K is homeomorphic to $\beta\omega$.*

In both cases we have a dichotomy result that basically characterizes two crucial properties about sets of continuous functions defined on a Polish and metric complete space, respectively. In this paper we look at this question in terms of the set of continuous functions $G \subseteq C(X, M)$ alone, when M is a general metric space. We first extend the notion of I_0-set, given by Hartman and Ryll-Nardzewski for complex-valued functions, to a more general setting, which will be needed later on when we apply it to topological groups.

Definition 1.3. Let X and M be a topological space and metric space, respectively. If $C(X, M)$ denotes the set of all continuous functions from X to M, we say that a subset Y of X is an *M-interpolation set* (or, we can simply say an *Interpolation set* for $C(X, M)$) when for each function $g \in M^Y$, which has relatively compact range in M, there exists a map $f \in C(X, M)$ such that $f_{|Y} = g$.

A main goal in this paper is the understanding of the key (topological) facts that characterize the existence of interpolation sets. Thereby, this research continues the task accomplished in previous projects [19,20] and [16]. Here, we introduce a crucial property stronger than the mere *non-equicontinuity*, that provides suffi cient conditions for the existence of Interpolation sets in diff erent settings. We refer to [6] forits motivation, where this notion implicitly appears.

Definition 1.4. Let X be a topological space and let M be a metric space. We say that $G \subseteq C(X, M)$ is a B-family if the following two conditions hold:

(a) G is relatively compact in M^X.
(b) There exists a nonempty open set V of X and $\epsilon > 0$ such that for every finite collection $\{U_1, \dots, U_n\}$ of nonempty relatively open sets of V there is $g \in G$ such that $\operatorname{diam}(g(U_j)) \geq \epsilon$ for all $j \in \{1, \dots, n\}$.

Remark 1.5. In [15], we define a subset G of $C(X, M)$ as *almost equicontinuous* (resp. *hereditarily almost equicontinuous*) if G is equicontinuous on a dense subset of X (resp. if G is almost equicontinuous for every closed nonempty subset of X). We do not know which is the relation between the notions of being

a B-family and the negation of being almost equicontinuous or hereditary almost equicontinuous when X is a Čech-complete space. However, in the cases in which this relation is known (topological groups, for instance), the existence of interpolation sets is assured as we show below.

Definition 1.6. A map $f : X \to Y$, defined between two topological spaces X and Y is *quasi-open* when for any open set U in X, the image $f(U)$ has nonempty interior.

We now formulate our main results. All topological spaces are assumed to be infinite, completely regular and Hausdorff from here on.

Theorem A. *Let X be a Čech-complete space, M a metric space, Y a metrizable separable space and $\Phi : X \to Y$ a continuous and quasi-open map. If $G \subseteq C(X, M)$ is a B-family such that each $g \in G$ factors through Φ (that is, for each $g \in G$, there is a map $\breve{g} \in C(Y, M)$ satisfying $g(x) = (\breve{g} \circ \Phi)(x)$ for all $x \in X$), then there is a nonempty compact subset Δ of X and a countable infinite subset L of G such that L is separated by Δ. As a consequence, if M is a Banach space, G contains a countably infinite M-interpolation set.*

Remark 1.8. From Theorem 1.2, one can deduce the existence of an interpolation subset in a set G of real-valued continuous functions defined on a complete metric space X, when $\overline{G}^{X}$ contains a function that is not Baire one. The main diff erence in our approach is that this property is isolated within the set G.

Theorem B. *Let X be a Čech-complete group and K a compact group. If $G \subseteq CHom(X, K)$ is not equicontinuous, then G contains a countable subset L such that $\overline{L}^{K^X}$ is canonically homeomorphic to βL, when L is equipped with the discrete topology. In case $K = \mathrm{U}(n)$, the unitary group of degree n, it follows that L is an $\mathbb{C}^{n^2}$-interpolation set.*

A consequence of this result is a variation of a well-known Theorem by Corson and Glicksberg [13] asserting that if a subset G of continuous homomorphisms defined on a hereditarily Baire group has a compact, metric closure, then it is equicontinuous. In case X is Čech-complete and K is a compact group, these constraints can be relaxed considerably.

Theorem C. *Let X be a Čech-complete group, K be a compact group and G be an infinite subset of $CHom(X, K)$. If for every countable subset $L \subseteq G$ and compact separable subset $Y \subseteq X$ we have that either $\overline{L}^{K^Y}$ has countable tightness or $|\overline{L}^{K^Y}| \leq \mathfrak{c}$, then G is equicontinuous.*

Definition 1.11. A Hausdorff topological space X is a *k_ω-space* if there exists an ascending sequence of compact subsets $K_1 \subseteq K_2 \subseteq \ldots \subseteq X$ such that $X = \underset{n<\omega}{\ } K_n$ and $U \subseteq X$ is open if and only if $U \cap K_n$ is open in K_n for each $n < \omega$ (i.e. $X = \varinjlim K_n$) as a topological space. A Hausdorff topological space X is *locally k_ω* if each point has an open neighborhood which is a k_ω-space in the induced topology. It is clear that every k_ω-space is a k-space (see [24]). A *k_ω-group* (resp. *locally k_ω-group*) is a topological group where the underlying topological space is a k_ω-space (resp. locally k_ω).

The class of abelian locally quasiconvex, locally k_ω-groups includes, in addition to all locally compact abelian groups: all free abelian groups on a compact space, indeed on any k_ω space; all dual groups of countable projective limits of metrizable (more generally, Čech-complete) abelian groups; all dual groups of abelian pro-Lie groups defined by countable systems [24,32]. Moreover, this class is preserved by countable direct sums, closed subgroups, and finite products [24].

Theorem D. *Let G be an abelian locally quasiconvex, locally k_ω-group. If $\{g_n\}_{n<\omega}$ is a sequence in G that is not precompact in G, then $\{g_n\}_{n<\omega}$ contains an I_0-set.*

The Bohr compactification of a topological group G, can be defined as a pair (bG, b) where bG is a compact Hausdorff group and b is a continuous homomorphism from G onto a dense subgroup of bG such that everycontinuous homomorphism $h\colon G \to K$ into a compact group K extends to a continuous homomorphism $h^b\colon bG \to K$, making the following diagram commutative:

$$\begin{tikzcd} G \arrow[rr, "b"] \arrow[dr, "h"'] & & bG \arrow[dl, "h^b"] \\ & K & \end{tikzcd}$$

The topology that b induces on G will be referred to as the *Bohr topology*. A topological group G is said to be *maximally almost periodic* (MAP, for short) when the map b is one-to-one, which implies that the Bohr topology will be Hausdorff .

The duality theory can be used to represent the Bohr compactification of an abelian group as a group of homomorphisms. Indeed, if G is an abelian topological group and Γ_d denotes its dual group equipped with the discrete topology then bG coincides with the dual group of Γ_d.

Given a topological group G, let G^+ denote the algebraic group G equipped with the Bohr topology. Glicksberg [23] has shown that in a locally compact abelian (LCA, for short) group G, every compact subset in G^+ is compact in G. This result concerning LCA groups is one of the pivotal results of the subject, often referred to as *Glicksberg's theorem*.

Given a topological group G and a property P, we say after Trigos-Arrieta [36] that G *respects the property* P when G and G^+ have the same sets satisfying P. Taking this terminology, Glicksberg's theorem asserts that locally compact Abelian groups respect compactness. Trigos-Arrieta considered some properties (pseudocompactness, countable compactness, functional boundedness) obtaining that they are respected by locally compact Abelian groups. Several authors have dealt with this question subsequently (cf. [3,5,28,18]).

Glicksberg result was extended in a diff erent direction by Comfort, Trigos-Arrieta and Wu [12] by the following remarkable result.

Let G be a LCA group and let N be a closed metrizable subgroup of its Bohr compactification bG. Denote by π the canonical projection from bG onto bG/N and set $b_N \overset{\mathrm{def}}{=} \pi \circ b$ making the following diagram commutative:

$$\begin{tikzcd} G \arrow[rr, "b"] \arrow[dr, "b_N"'] & & bG \arrow[dl, "\pi"] \\ & \frac{bG}{N} & \end{tikzcd}$$

Theorem 1.13 *(Comfort, Trigos-Arrieta and Wu). Let G be a LCA group and let N be a closed metrizable subgroup of its Bohr compactification bG. If A is a subset of G, then $A + (N \cap G)$ is compact in G if and only if the set $b_N(A)$ is compact in bG/N.*

In the same paper, the following classes of topological groups is introduced: A group G *strongly respects compactness* if satisfies the thesis in Theorem 1.13. The authors also propose the question of clarifying the relation between these two classes of groups and furthermore the characterization of the groups that strongly respect compactness. Using the techniques studied in this paper, we can prove that every abelian locally quasiconvex, locally k_ω group respects any compact-like property P that implies functional boundedness and, furthermore, strongly respects compactness, improving the results obtained by Gabriyelyan [18] for locally k_ω-groups. As a matter of fact, this result has been already applied to solve Question 4.1 in [12] (see [30]).

Definition 1.14. Let X be a topological space. A subset A of X is *functionally bounded* when every real-valued continuous function defined on X is bounded on A. We say that a topological property P on X is *stronger than or equal to functional boundedness* if for each $A \subseteq X$ that satisfies P ($A \sqsubset$ P for short), it holds that A is functionally bounded.

Theorem E. *Let G be an abelian, locally quasiconvex, locally k_ω group. Then the following holds:*

(i) G respects any compact-like property P stronger than or equal to functional boundedness.
(ii) G strongly respects compactness.

1. Interpolation sets in topological spaces

Definition 2.1. Let X and M be a topological space and metric space, respectively, and let $C(X, M)$ denote the space of continuous functions of X into M. Given a subset $L \subseteq C(X, M)$, we say that $K \subseteq X$ *separates* L if for every subset $A \subseteq L$ there are two closed subsets in M, say D_1 and D_2, and $x_A \in K$ such that $\mathrm{dist}(D_1, D_2) > 0$, $\chi(x_A) \in D_1$ for all $\chi \in A$ and $\chi(x_A) \in D_2$ for all $\chi \in L \setminus A$.

In the sequel, we are going to apply the definition of M-interpolation set to subsets $L \subseteq C(X, M) \subseteq M^X$, where X and M are a topological and a metric space, respectively. That is to say, we will look at L as an Interpolation set for $C(M^X, M)$. First, we need a lemma, whose proof is known. However, we include it here for the reader's sake. We refer to [14,21,34] for further information.

Lemma 2.2. *Let X and M be a topological and a metric space, respectively, and let L be a subset of $C(X, M)$ such that $\overline{L}^{M^X}$ is compact. Consider the following properties:*

(a) There is a nonempty subset Δ of X such that L is separated by Δ.
(b) Every two disjoint subsets of L have disjoint closures in M^X.
(c) $\overline{L}^{M^X}$ is canonically homeomorphic to βL if L is equipped with the discrete topology.
(d) L is a Interpolation set for $C(M^X, M)$.

Then $(a) \Rightarrow (b) \Leftrightarrow (c) \Leftarrow (d)$. If M is a Banach space then the properties (b), (c) and (d) are equivalent.

Proof. That (b) implies (c) is folklore. It is also clear that (d) implies (c). For (a) implies (b), let B_1 and B_2 two disjoint subsets of L, which is separated by Δ. Then, there are two closed sets D_1 and D_2 in M and $x_0 \in \Delta \subseteq X$ such that $d(D_1, D_2) \geq \epsilon_0$, for some $\epsilon_0 > 0$, $b_1(x_0) \in D_1$ for all $b_1 \in B_1$ and $\gamma(x_0) \in D_2$ for all $\gamma \in L \setminus B_1$ (in particular for all $b_2 \in B_2$). Thus, $\overline{B_1}^{M^X} \cap \overline{B_2}^{M^X} = \emptyset$. Finally, let us see that (c) implies (d), assuming that M is a Banach space.

Let $f \in M^L$ with relatively compact range in M. By (c), the map f can be extended to a continuous map defined on $\overline{L}^{M^X}$. Therefore, there is a continuous function $\overline{f}: \overline{L}^{M^X} \to M$ such that $\overline{f}_{|L} = f$. Now, applying [34, Cor. 3.5] (cf. [29, Th. 9]), it follows that there is a continuous map $\tilde{f}: M^X \to M$ that extends $\overline{f}$. Hence $\tilde{f}$ is the required extension of f to M^X. □

Definition 2.3. Let X and M be a topological space and a metric space (respectively) and let $f \in M^X$. We say that f is *totally discontinuous* if there are two subsets N_0 and N_1 in M and two dense subsets A_0 and A_1 in X such that $d(N_0, N_1) > 0$ and $f(A_j) \subseteq N_j$ for $j = 0, 1$.

We may assume that N_0 and N_1 are open sets because, otherwise, we would replace them by $B(N_i, s/3) \overset{\text{def}}{=} \{m \in M : d(m, N_i) < s/3\}$, where $s = d(N_0, N_1)$ and $i = 0, 1$.

Definition 2.4. A topological space X is said to be *Čech-complete* if it is a G_δ-subset of its Stone–Čech compatification. The family of Čech-complete spaces includes all complete metric spaces and all locally compact spaces.

Lemma 2.5. *Let X and M be a Čech-complete space and a metric space, respectively. If G a subset of $C(X, M)$ where each element has relatively compact range in M such that $\overline{G}^{M^X}$ contains a totally discontinuous function f, then there is a nonempty compact subset Δ of X and a countable infinite subset L of G, which is separated by Δ. Furthermore, by Lemma 2.2, if M is a Banach space, it follows that L is a M-interpolation set.*

Proof. Since X is Čech-complete, it is a G_δ-subset of its Stone–Cech compatification βX. Set $X = \bigcap_{n=0}^{\infty} W_n$, where W_n is a dense open subset of βX for each $n < \omega$ and $W_s \subseteq W_r$ if $r < s$. In the sequel, given a map $g \in C(X, M)$ with relatively compact range in M, we denote by g^β its continuous extension to βX.

Set N_0, N_1, A_0, A_1 as in Definition 2.3, where we assume that N_0 and N_1 are open wlog. By induction on $n = |t|$, $t \in 2^{(\omega)}$ (i.e. the set of finite sequences of 0's and 1's), we define a family $\{U_t : t \in 2^{(\omega)}\}$ of non-empty open subsets in βX and a sequence of functions $\{h_n : n < \omega\} \subseteq G$, satisfying the following conditions for all $t \in 2^{(\omega)}$:

(i) $U_\emptyset \subseteq \overline{U_\emptyset}^{\beta X} \subseteq W_0$;
(ii) $U_{ti} \subseteq \overline{U_{ti}}^{\beta X} \subseteq W_{|t|+1} \cap U_t$ for $i = 0, 1$;
(iii) $U_{t0} \cap U_{t1} = \emptyset$;
(iv) $h^\beta_{|t|}(U_{tj}) \subseteq N_j$ for $j = 0, 1$;
(v) if $s < |t|$, then $diam(h^\beta_s(\overline{U_{tj}}^{\beta X})) < \frac{1}{|t|}$ for $j = 0, 1$.

Construction: If $n = 0$, by regularity we can find $U_\emptyset$ a nonempty open set in βX such that $U_\emptyset \subseteq \overline{U_\emptyset}^{\beta X} \subseteq W_0$. For $n \geq 0$, suppose $\{U_t : |t| \leq n\}$ and $\{h_{|t|} : |t| < n\}$ have been defined satisfying $(i) - (v)$. Fix $t \in 2^{(\omega)}$ with $|t| = n$. Since U_t is open in βX, then $V_t \stackrel{\text{def}}{=} U_t \cap X \neq \emptyset$ is open in X. We can find $a_t, b_t \in V_t$ such that $f(a_t) \in N_0$ and $f(b_t) \in N_1$ because V_t is a relatively open subset of X and the sets A_0 and A_1 are dense in X. Since $f \in \overline{G}^{M^X}$, there is $h_n \in G$ such that $h_n(a_t) \in N_0$ and $h_n(b_t) \in N_1$.

Let h_n^β be the continuous extension of h_n, then we can select two open disjoint neighborhoods in βX, O_{t0} and O_{t1} of a_t and b_t, respectively, satisfying:

(1) $\overline{O_{t0}}^{\beta X} \cup \overline{O_{t1}}^{\beta X} \subseteq U_t$;
(2) $\overline{O_{t0}}^{\beta X} \cap \overline{O_{t1}}^{\beta X} = \emptyset$;
(3) $diam(h_n^\beta(O_{tj})) < \frac{1}{|t|}$;

(4) $h_n^\beta(O_{tj}) \subseteq N_j$, for $j = 0, 1$.

Since $W_{|t|+1}$ is dense in βX, then $W_{|t|+1} \cap O_{t0}$ and $W_{|t|+1} \cap O_{t1}$ are two nonempty open sets. By regularity, there exist two non empty open sets U_{t0} and U_{t1} such that $U_{t0} \subseteq \overline{U_{t0}}^{\beta X} \subseteq W_{|t|+1} \cap O_{t0}$ and $U_{t1} \subseteq \overline{U_{t1}}^{\beta X} \subseteq W_{|t|+1} \cap O_{t1}$, respectively. Therefore, U_{t0} and U_{t1} satisfies the conditions (ii), (iii) and (iv). Moreover, using a continuity argument, we can adjust the two open sets to satisfy (v).

Set $\Delta \stackrel{\text{def}}{=} \bigcap_{n<\omega} \bigcup_{|t|=n} \overline{U_t}^{\beta X}$, which is a closed subset of βX and, as a consequence, Δ is compact. On the other hand, we also have $\Delta = \bigcup_{\sigma \in 2^\omega} \bigcap_{n<\omega} \overline{U_{\sigma|n}}^{\beta X}$. For each $\sigma \in 2^\omega$, $\bigcap_{n<\omega} \overline{U_{\sigma|n}}^{\beta X} \neq \emptyset$ by compactness of βX. So $\Delta \neq \emptyset$. By construction we have that $\Delta \subseteq \bigcap_{n=0}^{\infty} W_n = X$. Consequently Δ is contained in X.

Define $\phi : \Delta \to 2^\omega$ by $\phi^{-1}(\sigma) = \bigcap_{n<\omega} \overline{U_{\sigma|n}}^{\beta X}$. Clearly ϕ is an onto and continuous map. For each $t \in 2^{(\omega)}$ and $\sigma \in 2^\omega$, $h_{|t|}(\phi^{-1}(\sigma))$ is a singleton by (v). Therefore, $h_{|t|}$ lifts to a continuous function $h^*_{|t|}$ on 2^ω such that $h_{|t|}(x) = h^*_{|t|}(\phi(x))$ for all $x \in \Delta$.

Let us see that $\{h_n\}_{n<\omega}$ is separated by Δ. Indeed, for any arbitrary subset $S \subseteq \omega$, it suffi ces to select $\sigma \in 2^\omega$ such that $\sigma(0) = 0$ and $\sigma(n+1) = 1$ if $n \in S$ or $\sigma(n+1) = 0$ if $n \notin S$. By construction, if we take any element $x \in \bigcap_{n<\omega} \overline{U_{\sigma|n}}^{\beta X} \subseteq \Delta$, then $h_n(x) \in N_1$ for every $n \in S$ and $h_n(x) \in N_0$ for every $n \notin S$.

Finally, in case M is a Banach space, Lemma 2.2 implies that $L = \{h_n\}_{n<\omega}$ is a M-interpolation set. □

We need the following compact space K that is defined as in [10].

Definition 2.6. Let (M, d) be a metric space that we always assume equipped with a bounded metric. We set

$$\mathrm{K} \stackrel{\text{def}}{=} \{\alpha : M \to [-1, 1] : |\alpha(m_1) - \alpha(m_2)| \leq d(m_1, m_2), \quad \forall m_1, m_2 \in M\}.$$

Being pointwise closed and equicontinuous by definition, it follows that K is a compact and metrizable subspace of $C(M, \mathbb{R})$ equipped with the supremum norm. For each $m_0 \in M$, we set $\alpha_{m_0} \in \mathbb{R}^M$ by $\alpha_{m_0}(m) \stackrel{\text{def}}{=} d(m, m_0)$ for all $m \in M$. It is easy to check that $\alpha_{m_0} \in \mathrm{K}$. Given any element $f \in M^X$, we associate a map $\check{f} \in \mathbb{R}^{X \times \mathrm{K}}$ defined by

$$\check{f}(x, \alpha) = \alpha(f(x)) \text{ for all } (x, \alpha) \in X \times \mathrm{K}.$$

In like manner, given any subset G of M^X we set $\check{G} \stackrel{\text{def}}{=} \{\check{f} : f \in G\}$.

We are now in position of proving Theorem A.

Proof of Theorem A. We may assume wlog that the map Φ is surjective because otherwise we would deal with the separable and metrizable space $\Phi(X)$. Due to the fact that Čech-completeness is hereditary for closed subsets, we may assume, from here on, that $X = \overline{V}$ wlog; where V is a nonempty open subset satisfying the following property: there is some fixed $\epsilon > 0$ such that for every finite collection $\{U_1, \ldots, U_n\}$ of non-empty open subsets contained in V, there is some element $g \in G$ with $diam(g(U_j)) \geq \epsilon$ for all $j \in \{1, \ldots, n\}$.

Let $\{\widetilde{V_k}\}_{k<\omega}$ be an arbitrary countable open basis in Y. We set $V_k \stackrel{\text{def}}{=} \Phi^{-1}(\widetilde{V_k})$ and pick and arbitrary point $x_k \in V_k$ for each $k < \omega$.

Since X is Čech-complete, there exists a sequence $\{\mathsf{A}_i\}_{i<\omega}$ of open coverings of X, such that, if a family F of closed subsets has the finite intersection property, and if for each $i < \omega$ there is an element of F such that is contained in a member of A_i, then $\mathsf{F} \neq \emptyset$ [14, Theorem 3.9.2]. In order to simplify the notation below, we say that a set of X is A_j-*small* if it is contained in a member of A_j.

Using an inductive argument, for every integer $n < \omega$, we find $f_n \in G$, $\alpha_n \in \mathsf{K}$ and a finite collection $\{U_{n,k}\}_{1\leq k\leq n}$ of nonempty open sets in X satisfying the following conditions (for each $n < \omega$ and each $k = 1, \ldots, n$):

(i) $U_{n,k} \subseteq V_k$;
(ii) $diam(f_n(U_{n,k})) \leq \frac{1}{n}$;
(iii) $\overline{U_{n+1,k}} \subseteq U_{n,k}$;
(iv) $d(f_n(x), f_n(x_k)) \geq \frac{\epsilon}{3}$ for all $x \in U_{n,k}$;
(v) $U_{n,k}$ is A_j-small, for $1 \leq j \leq n$;
(vi) $|\alpha_n(f_n(x)) - \alpha_n(f_n(x_k)| \geq \frac{\epsilon}{3}$, for all $x \in U_{n,k}$.

Construction: If $n = 1$, since V_1 is an open subset in X there exists $f_1 \in G$ such that $diam(f_1(V_1)) \geq \epsilon$. By the continuity of f_1, it follows that there exists a nonempty open subset $W_{1,1}$ such that:

(a) $W_{1,1} \subseteq V_1$
(b) $d(f_1(x), f_1(x_1)) \geq \frac{\epsilon}{3}$ for all $x \in W_{1,1}$

Let $\alpha_1 \stackrel{\text{def}}{=} \alpha_{f_1(x_1)} \in \mathsf{K}$. Note that $|\alpha_1(f_1(x)) - \alpha_1(f_1(x_1))| \geq \frac{\epsilon}{3}$ for all $x \in W_{1,1}$.

Now, we take the open covering A_1 of X. Then, there is $A \in \mathsf{A}_1$ such that $A \cap W_{1,1}$ is not empty. By regularity, we can find a nonempty open subset $U_{1,1}$ such that $U_{1,1} \subseteq \overline{U_{1,1}} \subseteq A \cap W_{1,1} \subseteq V_1$ and $diam(f_1(U_{1,1}) \leq 1$.

Assume now that f_n, α_n and $\{U_{n,k}\}_{1\leq k\leq n}$ have been obtained, with $n \geq 1$. By hypothesis, there exists $f_{n+1} \in G$ such that $diam(f_{n+1}(U_{n,k})) \geq \epsilon$ for all $k \in \{1, \ldots, n\}$ and $diam(f_{n+1}(V'_{n+1})) \geq \epsilon$, where $x_{n+1} \in V'_{n+1} \subseteq V_{n+1}$ and V'_{n+1} is A_j-small for $1 \leq j \leq n$.

By the continuity of f_{n+1}, we can find nonempty open subsets $\{W_{n+1,k}\}_{1\leq k\leq n+1}$ satisfying:

(1) $W_{n+1,k} \subseteq U_{n,k}$, for all $1 \leq k \leq n$;
(2) $W_{n+1,n+1} \subseteq V'_{n+1}$ (therefore $W_{n+1,n+1}$ is A_j-small for $1 \leq j \leq n$);
(3) $diam(f_{n+1}(W_{n+1,k})) \leq \frac{1}{n+1}$ for all $1 \leq k \leq n+1$;
(4) $d(f_{n+1}(x), f_{n+1}(x_k)) \geq \frac{\epsilon}{3}$ for all $x \in W_{n+1,k}$ and $1 \leq k \leq n+1$.

Set $\alpha_{n+1} \in [-1, 1]^M$ defined by

$$\alpha_{n+1}(m) \stackrel{\text{def}}{=} \min_{1\leq k\leq n+1} d(m, f_{n+1}(x_k)) \text{ for all } m \in M.$$

We claim that $\alpha_{n+1} \in K$. Indeed, if $m_1, m_2 \in M$, then

$$|\alpha_{n+1}(m_1) - \alpha_{n+1}(m_2)| = |\min_{1 \le k \le n+1} d(m_1, f_{n+1}(x_k)) - \min_{1 \le k \le n+1} d(m_2, f_{n+1}(x_k))|.$$

Assume wlog that

$$\min_{1 \le k \le n+1} d(m_1, f_{n+1}(x_k)) \ge \min_{1 \le k \le n+1} d(m_2, f_{n+1}(x_k))$$

and choose $k_0 \in \{1, \dots, n+1\}$ such that

$$\min_{1 \le k \le n+1} d(m_2, f_{n+1}(x_k)) = d(m_2, f_{n+1}(x_{k_0})).$$

Then,

$$|\alpha_{n+1}(m_1) - \alpha_{n+1}(m_2)| = \min_{1 \le k \le n+1} d(m_1, f_{n+1}(x_k)) - d(m_2, f_{n+1}(x_{k_0})) \le$$

$$d(m_1, f_{n+1}(x_{k_0})) - d(m_2, f_{n+1}(x_{k_0})) \le d(m_1, m_2).$$

On the other hand, for all $x \in W_{n+1,k'}$ and $1 \le k' \le n+1$:

$$|\alpha_{n+1}(f_{n+1}(x)) - \alpha_{n+1}(f_{n+1}(x_{k'}))| = |\alpha_{n+1}(f_{n+1}(x))| = \min_{1 \le k \le n+1} d(f_{n+1}(x), f_{n+1}(x_k)) \ge \frac{\epsilon}{3}.$$

Take the open covering $\mathcal{A}_{n+1}$ of X. Then, for each $k \in \{1, \dots, n+1\}$ there is $A_k \in \mathcal{A}_{n+1}$ such that $A_k \cap W_{n+1,k}$ is a nonempty open subset of X. By regularity we can find an open set $U_{n+1,k}$ such that:

- $U_{n+1,k} \subseteq \overline{U_{n+1,k}} \subseteq A_k \cap W_{n+1,k} \subseteq U_{n,k}$, if $1 \le k \le n$;
- $U_{n+1,n+1} \subseteq U_{n+1,n+1} \subseteq A_{n+1} \cap W_{n+1,n+1} \subseteq V_{n+1}$, if $k = n+1$.

This completes the construction.

Now, for each $k < \omega$, the intersection $\bigcap_{n=k}^{\infty} U_{n,k}$ is nonempty by Čech-completeness. Therefore, we can fix a point $\tilde{x}_k \in \bigcap_{n=k}^{\infty} U_{n,k}$ for all $k < \omega$. Note that $\Phi(x_k) \in \tilde{V}_k$ and $\Phi(\tilde{x}_k) \in \tilde{V}_k$ for all $k \in \omega$.

Take an element $(f, \alpha) \in \overline{\{(f_n, \alpha_n)\}_{n<\omega}}^{(C_M^X \times K)}$. By (vi) we have:

$$|\alpha_n \circ \tilde{f}_n(\Phi(\tilde{x}_k)) - \alpha_n \circ \tilde{f}_n(\Phi(x_k))| = |\alpha_n \circ f_n(\tilde{x}_k) - \alpha_n \circ f_n(x_k)| \ge \frac{\epsilon}{3}, \quad \forall n \ge k.$$

Therefore, $osc(\alpha_n \circ \tilde{f}_n, \tilde{V}_k) \ge \frac{\epsilon}{3}$ for all $n \ge k$. As a consequence, we also have $osc(\alpha \circ \tilde{f}, \tilde{V}_k) \ge \frac{\epsilon}{3}$ for all $k < \omega$.

Let $\{r_m, \delta_m\}_{m<\omega}$ be an enumeration of all pairs of rational numbers (r, δ) with $\delta > 0$. For each $m < \omega$, define

$$\tilde{F}_m = \{y \in Y : \inf(\alpha \circ \tilde{f})(U) < r_m, \quad \sup(\alpha \circ \tilde{f})(U) \ge r_m + \delta_m, \forall \text{nbd } U \text{ of } y\}.$$

It is easily seen that $\tilde{F}_m$ is closed and, consequently, $F_m \stackrel{\text{def}}{=} \Phi^{-1}(\tilde{F}_m)$ is closed in X.

Observe that, since $\{\tilde{V}_k\}_{k<\omega}$ is an open basis in Y, it follows that $Y = \bigcup_{m<\omega} \tilde{F}_m$ and, hence $X = \bigcup_{m<\omega} F_m$. Being X Čech-complete, it is a Baire space. Therefore, there is some $m_0 < \omega$ such that F_{m_0} has nonempty interior in . Since Φ is a quasi-open map, we have that $\overline{\Phi(\ )}$ has nonempty interior included in . It follows that $\inf\ \alpha \circ \tilde{f}(\tilde{U}) < r_{m_0}$ and $\sup\ \alpha \circ \tilde{f}(\tilde{U}) \geq r_{m_0} + \delta_{m_0}$. Set $U_0 = \Phi^{-1}(U) \subseteq U$ we have that $\inf(\alpha \circ f(U_0)) < r_{m_0}$ and $\sup(\alpha \circ f(U_0)) \geq r_{m_0} + \delta_{m_0}$.

Set $F = \overline{U_0}$, $r \stackrel{\mathrm{def}}{=} r_{m_0}$ and $\delta \stackrel{\mathrm{def}}{=} \delta_{m_0}$ and consider the following sets:

$$A_0 = \{x \in F : \alpha \circ f(x) < r\} = \{x \in F : \alpha \circ f(x) \in I_0\}$$
$$A_1 = \{x \in F : \alpha \circ f(x) \geq r + \delta\} = \{x \in F : \alpha \circ f(x) \in I_1\}$$

where $I_0 = [-1, r)$ and $I_1 = (r+\delta, 1]$. Note that A_0 and A_1 are dense subsets in F. Define $N_0 \stackrel{\mathrm{def}}{=} \alpha^{-1}(I_0)$ and $N_1 \stackrel{\mathrm{def}}{=} \alpha^{-1}(I_1)$, which are disjoints. Moreover, since $\alpha \in K$, it follows that $d(N_0, N_1) \geq \delta$ and $f(A_j) \subseteq N_j$ for $j = 0, 1$. Therefore f is totally discontinuous on F. It now suffices to apply Lemma 2.5. □

Remark 2.7. Note that the result remains valid if we assume that for each residual subset R of X there is a separable metrizable space Y and a continuous and quasi-open map $\Phi : R \to Y$ such that for all $g \in G$ there is a $g \in C(Y, M)$ satisfying $g(x) = (\tilde{g} \circ \Phi)(x)$ for all $x \in R$.

Corollary 2.8. *Let X be a Polish space, let (M, d) be a metric space and let $G \subseteq C(X, M)$ be a B-family. Then there is a nonempty compact subset Δ of X and a countable subset L of G such that L is separated by Δ. As a consequence, if M is a Banach space, it follows that L is a M-interpolation set.*

2. Interpolation sets in topological groups

In this section, we apply the results obtained previously in the setting of topological groups. Our first result clarifies the relevance of the notion of B-family when we deal with topological groups. From here on, we assume, wlog, that every metrizable topological group M is equipped with a left-invariant metric. Furthermore, if M is in addition compact, then we assume that M is equipped with a bi-invariant metric. From here on, if X and Y are topological groups, we let $Hom(X, M)$ (resp. $CHom(X, M)$) denote the set of all homomorphisms (resp. continuous homomorphisms) of X into M.

Lemma 3.1. *Let X be a topological group, M a metric topological group and $G \subseteq CHom(X, M)$ such that $\overline{G}^{M^X}$ is compact. Then G is a B-family if and only if it is not equicontinuous.*

Proof. It is clear that, if G is a B-family, then it may not be equicontinuous. So, assume that G is not a B-family. Taking $V = X$ and $\epsilon > 0$ arbitrary, there exists a finite family $U_1, \ldots, U_n$ open subsets in X (wlog, we assume that $U_j = x_j V_j$, where V_j is a neighborhood of the neutral element) such that for every $g \in G$ there is V_j, with $1 \leq j \leq n$, satisfying that $diam(g(x_j V_j)) < \epsilon$. Now, since g is a group homomorphism and d is left-invariant, it follows that $diam(g(V_j)) < \epsilon$ as well. Set $V_0 = V_1 \cap \ldots \cap V_n$, then $diam(g(xV_0)) < \epsilon$ for all $g \in G$ and $x \in X$. Consequently G is equicontinuous. □

The next result is a direct consequence of Lemma 3.1, Theorem A and Lemma 2.2. Previously, we need the following definition. Recall that $\mathrm{U}(n)$ denotes the unitary group of degree n.

Corollary 3.2. *Let X be a compact group, M a metric topological group and $G \subseteq CHom(X, M)$ such that $\overline{G}^{M^X}$ is compact. If G is not equicontinuous, then there is a nonempty compact subset Δ of X and a countable infinite subset L of G that is separated by Δ. As a consequence, if M is a Banach space, it follows that L is a M-interpolation set. In particular, if $M = \mathrm{U}(n)$ then G contains an Interpolation set for $C(Hom(G, \mathrm{U}(n)), \mathbb{C}^{n^2})$.*

Proof. By Troallic [37], we may assume wlog that G is countable. By Lemma 3.1, G is a B-family. Define an equivalence relation on X by $x \sim y$ if and only if $g(x) = g(y)$ for all $g \in G$. Since G is countable and consists of group homomorphisms, it follows that the quotient space $\tilde{X} = X/\sim$ is a compact metrizable group. Therefore, if $p : X \to \tilde{X}$ denotes the canonical quotient map, each $g \in G$ factors through a map $\tilde{g}$ defined on $CHom(\tilde{X}, M)$; that is $\tilde{g}(p(x)) \stackrel{def}{=} g(x)$ for any $x \in X$. Since every quotient group homomorphism is automatically open, Theorem A implies that there is a nonempty subset Δ of X and a subset L of G such that L is separated by Δ. In case M is a Banach space, applying Lemma 2.2, we obtain that L is a M-interpolation set. □

Next result is folklore but we include its proof for the sake of completeness.

Lemma 3.3. *Let X be a topological group, M a metric topological group, $G \subseteq C(X, M)$ and $h \in C(X, M)$. Set $Gh \stackrel{def}{=} \{gh : g \in G\}$. Then G is equicontinuous on X if and only if Gh is equicontinuous on X.*

Proof. It suffices to prove that Gh is equicontinuous if G is equicontinuous. Let x_0 be an arbitrary but fixed point in X. Since right translations are continuous mappings on a topological group, and G (resp. h) is equicontinuous (resp. continuous) on X, given $\epsilon > 0$, there is a neighborhood U of x_0 such that $d(g(x_0)h(x_0), g(x)h(x_0)) < \epsilon/2$ and $d(h(x_0), h(x)) < \epsilon/2$ for all $x \in U$ and all $g \in G$. Thus, applying the left invariance of the group metric, we obtain

$$d(g(x_0)h(x_0), g(x)h(x)) \leq d(g(x_0)h(x_0), g(x)h(x_0)) + d(g(x)h(x_0), g(x)h(x)) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

for all $x \in U$, which completes the proof. □

With the hypothesis of the previous lemma, if $g \in CHom(X, M)$, the symbol g^{-1} denotes the map defined by $g^{-1}(x) = g(x)^{-1} = g(x^{-1})$ for all $x \in X$. Combining Lemmata 3.1 and 3.3, we obtain:

Corollary 3.4. *Let X be a topological group, M be a topological group with a bi-invariant metric, $G \subseteq CHom(X, M)$ such that $\overline{G}^{M^X}$ is compact and $g_0 \in G$. Then Gg_0^{-1} is a B-family if and only if it is not equicontinuous.*

Proof. It suffices to see that Gg_0^{-1} is equicontinuous if Gg_0^{-1} is not a B-family. Reasoning as in Lemma 3.3, let $V = X$ and $\epsilon > 0$, then there are $\{U_1, \ldots, U_n\}$ open subsets of X such that for all $g \in G$ there is $j \in \{1, \ldots, n\}$ with $diam(gg_0^{-1}(U_j)) < \epsilon$. We can assume that $U_j = x_jV_j$ wlog, where V_j is a neighborhood of the identity element of X, for all $1 \leq j \leq n$. Take $W \stackrel{def}{=} \bigcap_{1 \leq j \leq n} V_j$ and an arbitrary element $x_0 \in X$.

Given $g \in G$, there is $j \in \{1, \ldots, n\}$ such that

$$\begin{aligned}
\epsilon > diam(gg_0^{-1}(U_j)) = diam(gg_0^{-1}(x_jV_j)) &= \sup_{x,y \in V_j} d(gg_0^{-1}(x_jx), gg_0^{-1}(x_jy)) \\
&= \sup_{x,y \in V_j} d(g(x_j)\ gg_0^{-1}(x)\ g_0^{-1}(x_j), g(x_j)\ gg_0^{-1}(y)\ g_0^{-1}(x_j)) \\
&= \sup_{x,y \in V_j} d(\ gg_0^{-1}(x)\ ,\ gg_0^{-1}(y)\ ) = diam(gg_0^{-1}(V_j)) \\
&\geq diam(gg_0^{-1}(W)) = diam(gg_0^{-1}(x_0W)). \quad \square
\end{aligned}$$

We can now prove Theorem B.

Proof of Theorem B. Since K is compact, there is a representation $\pi : K \to \mathrm{U}(n)$ such that $\{\pi \circ g : g \in G\}$ is not equicontinuous. Therefore, we assume that $K = \mathrm{U}(n)$ wlog.

Applying [15, Cor. 2.4] (cf. [37, Cor. 3.2]), since $G \subseteq CHom(X, \mathrm{U}(n))$ is not equicontinuous, there exists a separable compact subset F of X and a countable subset $L \subseteq G$ such that $L|_F$ is not equicontinuous. Set H as the smallest closed subgroup generated by F, it follows that $H \leq X$ is closed and separable and $L \subsetneqq$ countable such that L_H is not equicontinuous. So we can assume wlog that X is separable and G is countable. On the other hand, by Čech-completeness of X, there must be a compact subgroup C of X such that X/C is separable, complete and metrizable [8], thereby, a Polish space.

Let $G|_C \overset{\text{def}}{=} \{g|_C : g \in G\} \subseteq CHom(C, \mathrm{U}(n))$. We have two possible cases:

(1) $G|_C$ contains infinitely many elements that are pairwise inequivalent (recall that $\gamma_1, \gamma_2 \in Hom(C, \mathrm{U}(n))$ are equivalent ($\gamma_1 \sim \gamma_2$) if there exists $U \in \mathrm{U}(n)$ such that $\gamma_1 = U^{-1}\gamma_2 U$).
(2) $G|_C$ only contains a finite subset of elements that are pairwise inequivalent.

- Case (1): We may suppose wlog that all elements of $G|_C$ are pairwise inequivalent. By [11, Th. 1], it follows that $G|_C$ is discrete as a subset of $CHom(C, \mathrm{U}(n))$ in the compact open topology on C, which implies that $G|_C$ may not be equicontinuous on C. Applying Corollary 3.2, there is a nonempty subset Δ of C and a countable subset L of G such that L is separated by Δ. Thus, by Lemma 2.2, $\overline{L}^{\mathrm{U}(}$ is canonically homeomorphic to βL (where L is equipped with the discrete topology) and we are done.
- Case (2): Set $H \overset{\text{def}}{=} \{\phi_1, \ldots, \phi_m\} \subseteq G$ such that every $g \in G$ is equivalent to an element in H when they are restricted to C. If we define $G_i = \{g \in G : g|_C \sim \phi_i|_C\}$, then $G = G_1 \ \ldots \ G_m$. Since G is not equicontinuous, there is $i \in \{1, \ldots m\}$ such that G_i is not equicontinuous. So, we may assume wlog that there is $g_0 \in G$ such that $g|_C \sim g_0|_C$ for all $g \in G$. Therefore, for each $g \in G$, there is $U_g \in \mathrm{U}(n)$ with $(U_g^{-1} g U_g)|_C = g_0|_C$. Denote by $\tilde{g}$ the map $U_g^{-1} g U_g$ and set $\tilde{G} \overset{\text{def}}{=} \{U_g^{-1} g U_g : g \in G\}$, which is a subset of $CHom(X, \mathrm{U}(n))$. It is easily seen that $\tilde{G}$ is not equicontinuous on X. (Indeed, assume that $\tilde{G}$ were equicontinuous and let W be an open neighborhood of the identity matrix I_n in $\mathrm{U}(n)$. By [31, Corollary 1.12], there would exist an open neighborhood V of e_X such that $\tilde{g}(V) \subseteq \bigcap_{U \in \mathrm{U}(n)} U^{-1} W U$ for

all $\tilde{g} \in \tilde{G}$. Therefore, we would have $g(v) = U_g \tilde{g}(v) U_g^{-1} \in W$ for all $v \in V$. This would imply that G is equicontinuous, which is a contradiction.) Hence, $\tilde{G} g_0^{-1}$ is a B-family on X by Lemmas 3.3 and 3.4. Let $\pi_C : X \to X/C$ the canonical quotient map, which is open and continuous. Since X/C is Polish and each $g g_0^{-1}$ factors through X/C, we apply Theorem A and Lemma 2.2 in order to obtain $\Delta \subseteq X$ and $\tilde{L} \subseteq \tilde{G}$ such that

$$\overline{\tilde{L}}^{\mathrm{U}(n)^{\Delta}} \quad \overline{\tilde{L}}^{\mathrm{U}(n)^{\Delta}} g_0^{-1} \quad \beta \tilde{L}.$$

Set $L \overset{\text{def}}{=} \{g : \tilde{g} \in \tilde{L}\} \subseteq G$ and consider the map

$$\begin{aligned} \psi : (\tilde{L}, t_p(\Delta)) &\longrightarrow (L, t_p(\Delta)) \\ U_g^{-1} g U_g &\longmapsto g \end{aligned}$$

The map ψ is continuous because $\tilde{L}$ is discrete. Moreover, using that $\overline{\tilde{L}}^{\mathrm{U}(n)^X}$ is canonically homeomorphic to βL (L with the discrete topology), there is a continuous extension map

$$\overline{\psi} : (\overline{\tilde{L}}^{\mathrm{U}(n)^X}, t_p(\Delta)) \to (\overline{L}^{\mathrm{U}(n)^X}, t_p(\Delta)).$$

Using a compactness argument on the group $U(n)$, it is not hard to verify that if $p, q \in \tilde{L}^{U(n)^X}$ and $\overline{\psi}(p) = \overline{\psi}(q)$ then p and q are equivalent. Since $Orbit(p) = \{ U^{-1}pU : U \in U(n)\}$ has the cardinality of continuum c and $|\tilde{\overline{L}}^{U(n)^X}| = |\beta L| = |\beta\omega| = 2^c$, we obtain:

$$2^c = |\tilde{\overline{L}}^{U(n)^X}| \leq |\overline{L}^{U(n)^X}||U(n)| = \max\{|\overline{L}^{U(}|, c\}.$$

Therefore

$$|\overline{L}^{U(n)^X}| \geq 2^c.$$

Applying [16, Cor. 2.16], it follows that L contains a subset P such that $\overline{P}^{U(n)^X}$ is canonically homeomorphic to βP (with P equipped with the discrete topology). This completes the proof. □

Corollary 3.5. *Let X be a Čech-complete abelian group. If an infinite subset G of $\hat{X}$ is not equicontinuous, then G contains a countably infinite I_0-set.*

Next follows the proof of Theorem C.

Proof of Theorem C. If for every countable subset $L \subseteq G$ and compact separable subset $Y \subseteq X$ we have that either $\overline{L}^{K^Y}$ has countable tightness or $|\overline{L}^{K^Y}| \leq c$, then $\overline{L}^{K^Y}$ may not contain any copy of $\beta\omega$. By Theorem B, this implies that $L_{|Y}$ is equicontinuous on Y. Applying [15, Theorem B], it follows that G is hereditarily equicontinuous on X, which implies that G is equicontinuous because G consists of group homomorphisms. □

3. I_0-sets in abelian locally k_ω groups

In this section, we study the existence of I_0-sets for abelian locally k_ω groups, a large family of topological groups that includes, for example, all LCA groups, the free abelian groups on a compact space and all countable direct sum of compact groups. The proof of our main results are obtained using methods of Pontryagin–van Kampen duality. Therefore, we first recall some basic definitions and facts about the Pontryagin duality of abelian groups. From here on, all groups are supposed to be abelian and, therefore, we will use additive notation to deal with them. In particular, we identify $\mathbb{T}$ with the additive group $[-1/2, 1/2)$ by identifying $\pm 1/2$.

If G is a topological group, *a character* on a topological abelian group G is a continuous group homomorphism from G to the torus group $\mathbb{T}$. The set of all characters on G, with pointwise addition, is a group.

For a topological abelian group G, let $\mathrm{K}(G)$ denote the family of all compact subsets of G. For a set $A \subseteq G$ and a positive real ϵ, define

$$[A, \epsilon] \stackrel{\text{def}}{=} \{\chi \in \hat{G} : |\chi(a)| \leq \epsilon \text{ for all } a \in A\}.$$

The sets $[K, \epsilon] \subseteq \hat{G}$, for $K \in \mathrm{K}(G)$ and $\epsilon > 0$, form a neighborhood base at the trivial character, defining the compact-open topology. We write $\hat{G}$ for the topological abelian group obtained in this manner.

A topological abelian group G is *reflexive* if the evaluation map E

$$E : G \to \hat{\hat{G}},$$

defined by $E(g)(\chi) = \chi(g)$ for all $g \in G$ and $\chi \in \hat{G}$, is a topological isomorphism. By the Pontryagin–van Kampen theory, we know that every LCA group is reflexive. Furthermore, the dual of a compact group is discrete and the dual of a discrete group is compact. In general, the dual of a LCA group is also locally compact. As a consequence, every compact abelian group is equipped with the topology of pointwise convergence on its dual group.

Definition 4.1. Let G be a topological abelian group. For $A \subseteq G$, let $A^D := [A, 1/4]$. Similarly, for $X \subseteq \hat{G}$, let

$$X^D := \left\{ g \in G : |\chi(g)| \leq \frac{1}{4} \text{ for all } \chi \in X \right\}.$$

The following facts are well known (see [4]).

Lemma 4.2. *For each neighborhood U of 0 in G, we have that $U^D \in K(\hat{G})$.*

Definition 4.3. Let G be a topological abelian group. A set $A \subseteq G$ is *quasiconvex* if $A^{DD} = A$. The topological group G is *locally quasiconvex* if it has a neighborhood base at the neutral element, consisting of quasiconvex sets. G is called *Maximally Almost Periodic* (MAP, for short) in the sense of von Neumann when $\hat{G}$ separates the points in G, which means that G^+ is Hausdorff and, as a consequence, the group G is algebraically injected in its Bohr compactification bG. Obviously, by definition, every locally quasiconvex abelian group G is MAP.

For each set $A \subseteq G$, the set A^D is a quasiconvex subset of $\hat{G}$. Thus, the topological group $\hat{G}$ is locally quasiconvex for each topological abelian group G. Moreover, local quasiconvexity is hereditary for arbitrary subgroups.

The set A^{DD} is the smallest closed, quasiconvex subset of G containing A.

In the case where G is a topological vector space, G is locally quasiconvex in the present sense if, and only if, G is a locally convex topological vector space in the ordinary sense.

If G is locally quasiconvex, its characters separate points of G, and thus the evaluation map $E : G \to \hat{\hat{G}}$ is injective. For each quasiconvex neighborhood U of 0 in G, the set U^D is a compact subset of $\hat{G}$ (Lemma 4.2), and thus U^{DD} is a neighborhood of 0 in $\hat{\hat{G}}$. As $E[G] \cap U^{DD} = E[U^{DD}] = E[U]$, we have that E is open [4, Lemma 14.3].

The following theorem of Glöckner, Gramlich and Hartnick [24] states that there exists a relation between the abelian locally k_ω groups and the abelian Čech-complete groups.

Theorem 4.4. *If G is an abelian locally k_ω group, the $\hat{G}$ is Čech-complete.*
Conversely, $\hat{G}$ is locally k_ω, for each abelian Čech-complete topological group G.

Using this duality and Theorem B we obtain:

Proof of Theorem D. Consider the abelian Čech-complete group $\hat{G}$. By means of the evaluation map $E : G \to \hat{\hat{G}} \subseteq C(\hat{G}, \mathbb{T})$, we can look at the sequence $\{g_n\}_{n<\omega}$ as a subset of $C(\hat{G}, \mathbb{T})$. Furthermore, since $\{g_n\}_{n<\omega}$ is not precompact in G, it follows that $\{g_n\}_{n<\omega}$ is not equicontinuous on $\hat{G}$. Indeed, if it were equicontinuous on $\hat{G}$, by Arzelà–Ascoli's theorem, then $\{g_n\}_{n<\omega}$ would be precompact in $C_c(\hat{G}, \mathbb{T})$, the group $C(\hat{G}, \mathbb{T})$ equipped with the compact open topology. Now, since G is a locally quasiconvex k-space, the evaluation map $E : G \to \hat{\hat{G}}$ is a topological isomorphism onto its image (see [27]). Thus $\{g_n\}_{n<\omega}$ would also be precompact in G, which is a contradiction.

Therefore, the sequence $\{g_n\}_{n<\omega}$ is not an equicontinuous set on $\hat{G}$ and, by Theorem B, contains an I_0-set. □

The next result was proved in [19, Lemma 4.11].

Lemma 4.5. *Let G be a maximally almost periodic abelian group, A a subset of G and let N be a subset of bG containing the neutral element such that $A + N$ is compact in bG. If F is an arbitrary subset of A, there exists $A_0 \subseteq A$ with $|A_0| \leq |N|$ such that*

$$cl_{bG}F \subseteq A_0 + N + cl_{G^+}(F - F).$$

We are now in position of proving Theorem E.

Proof of Theorem E. Let G be a locally quasiconvex, locally k_ω group and let bG denote its Bohr compactification.

(i) Let P be a topological property implying functional boundedness and let A be any subset of G satisfying P in G^+, which (by definition) is equipped with the weak topology generated by $\hat{G}$; that is to say $G^+ \subseteq \mathbb{T}^{\hat{G}}$. Reasoning by contradiction assume that A does not satisfy P in G. We claim that A may not be a precompact subset of G. Indeed, if it were, since every locally k_ω-group is complete [24, Remark 7.3], it would follow that $\overline{A}^G$ would be compact in G. Therefore, it would also be compact in G^+ that is equipped with a weaker topology. Since any compact topology is (Hausdorff) minimal, this would imply that the Bohr topology would coincide with the original topology of G on the compact subset $\overline{A}^G$ and, as a consequence, on A. Thus A would have property P in G.

So, assume wlog that A is not precompact in G. As in the proof of Theorem D, if we take the abelian Čech-complete group $\hat{G}$ and inject G in $C(\hat{G}, \mathbb{T})$ by means of the evaluation map $E : G \to \hat{\hat{G}} \subseteq C(\hat{G}, \mathbb{T})$, it follows that A is not equicontinuous on $\hat{G}$. By [15, Cor. 2.4], it follows that there exists a countable subset $F \subseteq A$ and a separable compact subset $X \subseteq \hat{G}$ such that F is not equicontinuous on X. Taking the closure in $\hat{G}$ of the subgroup generated by X, we may assume wlog that X is a separable closed subgroup of $\hat{G}$. On the other hand, since A is functionally bounded in G^+ and $X \subseteq \hat{G}$, it follows that F is also functionally bounded in G, when the latter is equipped with weak topology generated by X.

Set $X^\perp \overset{def}{=} \{g \in G : \chi(g) = 0 \text{ for all } \chi \in X\}$ and take the quotient $G/X^\perp$, which clearly is a maximally almost periodic group whose dual is X. Furthermore, the group $G/X^\perp$ is locally k_ω and $\widehat{G/X^\perp} \cong X$, which is Čech-complete. If $p : G \to G/X^\perp$ denotes the open quotient map and $bG/X^\perp$ denotes the Bohr compactification of $G/X^\perp$, it follows that there is a canonical extension $p^b : bG \to bG/X^\perp$. Therefore, we have that $p^b(F)$ is a functionally bounded subset of $p^b(G^+) = G^+/X^\perp$ that is not equicontinuous on X. Indeed, if $p^b(F)$ were equicontinuous on X, then it would follow that F would be equicontinuous on X, which is not true. In other words, we may assume wlog that A is a countable, functionally bounded subset of G^+ that is not equicontinous on $\hat{G}$, which is separable.

As in the proof of Theorem B, by the Čech-completeness of X, there must be a compact subgroup C of X such that X/C is separable, complete and metrizable [8], thereby, a Polish space.

Let $A|_C = \{g|_C : g \in A\} \subseteq \hat{C}$. We have two possible cases:

(1) $A_{|C}$ contains infinitely many diff erent elements.
(2) $A_{|C}$ only contains a finite number of diff erent elements.

- Case (1): By Pontryagin duality, the dual of a compact group (equipped with the compact open topology) is discrete. Therefore $A_{|C}$ is an infinite discrete subset of $\hat{C}$ in the compact open topology on C, which implies that $A_{|C}$ may not be equicontinuous on C. Applying Corollary 3.2, there is a nonempty subset Δ of C and a countable subset L of A such that L is separated by Δ. Thus, by Lemma 2.2, $\overline{L_{|\Delta}}^{\mathbb{T}^\Delta}$ is canonically homeomorphic to βL (where L is equipped with the discrete topology), which yields

$$|\overline{L_{|\Delta}}^{T^{\Delta}}| = 2^{c}. \qquad \text{(I)}$$

On the other hand, by Asanov and Velichko's generalization of a well-known theorem of Grothendieck about sets of continous functions defined on a compact space [2] (see also [1, III.4.1.]), we have that $C_p(\Delta, T)$ (the space of continous functions of Δ into T equipped with the pointwise convergence topology) is a μ-space, which means that every closed functional bounded subset of $C_p(\Delta, T)$ is compact. Being A a functionally bounded subset of G^+, which is equipped with the pointwise convergence topology on $\hat{G} \supseteq \Delta$, it follows that $L_{|\Delta} A_{|\Delta}$ is functionally bounded as a subset of $C_p(\Delta, T)$. Thus $\overline{L_{|\Delta}}^{C_p(\Delta,T)}$ must be compact and, as a consequence

$$\overline{L_{|\Delta}}^{C_p(\Delta,T)} = \overline{L_{|\Delta}}^{T^{\Delta}}.$$

However, by [20, Lemma 3.4], we have that $|\overline{L_{|\Delta}}^{C_p(\Delta,T)}| \leq c$, which is in contradiction with (I).

- Case (2): Set $H \stackrel{\text{def}}{=} \{\phi_1, \ldots, \phi_m\} \subseteq A$ such that for each $g \in A$ there is an element $\phi_i \in H$ satisfying that $g_{|C} = \phi_{i|C}$. If we define $A_i = \{g \in A : g_{|C} = \phi_{i|C}\}$, then $A = A_1 \cup \ldots \cup A_m$. Since A is not equicontinuous, there is $i \in \{1, \ldots m\}$ such that A_i is not equicontinuous. So, we may assume wlog that there is $g_0 \in A$ such that $g_{|C} = g_{0|C}$ for all $g \in A$. By Lemma 3.3, we know that Ag_0^{-1} is not equicontinuous on X and, by Lemma 3.4, it follows that Ag_0^{-1} is a B-family on X.

Let $\pi_C : X \to X/C$ the canonical quotient map, which is open and continuous. Since X/C is Polish and each gg_0^{-1} factors through X/C, we apply Theorem A and Lemma 2.2 in order to obtain $\Delta \subseteq X$ and $L \subseteq A$ such that

$$\overline{L_{|\Delta}}^{T^{\Delta}} \quad \overline{L_{|\Delta} g_0^{-1}}^{T^{\Delta}} \quad \beta L,$$

which again yields

$$|\overline{L_{|\Delta}}^{T^{\Delta}}| = 2^{c}. \qquad \text{(II)}$$

On the other hand, repeating the same argument as in (1), we deduce that $|\overline{L_{|\Delta}}^{T^{\Delta}}| = |\overline{L_{|\Delta}}^{C_p(\Delta,T)}| \leq c$, which again is in contradiction with (II). This completes the proof of (i).

(ii) The proof of this part replays some of the steps followed to prove (i). For the reader's sake, we will avoid unnecessary repetitions as much as possible.

Let N be a closed metrizable subgroup of bG and let A be a subset G. It is obvious that if $A + (N \cap G)$ is compact in G, then $b_N(A)$ is compact in bG/N.

In order to prove the non-trivial converse implication, again reasoning by contradiction, assume that $b_N(A)$ is compact in bG/N but $A + (N \cap G)$ is not compact in G. As G is complete [24, Remark 7.3], this means that $A + (N \cap G)$, being closed in G, is not precompact in the topology inherited from G. As in the proof of (i), it follows that there exists a countable subset $F \subseteq A + (N \cap G)$ and a separable compact subset $X \subseteq \hat{G}$ such that F is not equicontinuous on X. Taking the closure in $\hat{G}$ of the subgroup generated by X, we may assume that X is a separable closed subgroup of $\hat{G}$.

Again, the quotient group $G/X^{\perp}$ is a MAP, locally k_ω group whose dual group X is Čech-complete. If $p: G \to G/X^{\perp}$ denotes the open quotient map and $p^b: bG \to bG/X^{\perp}$ is the canonical extension to their Bohr compactifications, we have that $p(A + (N \cap G))$ is contained in $p^b(A + N) = p(A) + p^b(N)$, which is compact in $bG/X^{\perp}$. Applying Lemma 4.5 to $p(F)$ and $p^b(N)$, we obtain that there exists $A_0 \subseteq p(A)$ with $|A_0| \leq |p^b(N)| \leq c$ such that

$$cl_{bG/X^{\perp}}p(F) \subseteq A_0 + p^b(N) + cl_{(G/X^{\perp})^+}p(F - F).$$

Now, being the group X separable, it follows that $G/X^{\perp}$ can be equipped with a metrizable precompact topology. As a consequence $|G/X^{\perp}| \leq \mathfrak{c}$. All in all, we obtain that $|cl_{bG/X^{\perp}}p(F)| \leq \mathfrak{c}$.

On the other hand, $p(F)$ is not equicontinuous as a subset of $C(X, \mathbb{T})$ and, by Theorem D, this means that it contains an I_0-set, which yields $|cl_{bG/X^{\perp}}p(F)| = |\beta\omega| = 2^{\mathfrak{c}} > \mathfrak{c}$. This is a contradiction that completes the proof. □

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