ISSN (e): 2250 – 3005 || Volume, 08 || Issue, 10|| October – 2018 ||
International Journal of Computational Engineering Research (IJCER)

Magnetic curves in tangent sphere bundles II

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Abstract: *We study contact magnetic curves in the unit tangent sphere bundle over the Euclidean plane. In particular, we obtain all contact magnetic curves which are slant.*
Keywords: *Magnetic field Tangent sphere bundle α-Sasakian manifold Periodic curve*

I. INTRODUCTION AND PRELIMINARIES

As is well known, unit tangent sphere bundle over Riemannian manifolds admits the so-called standard contact metric structure. In our previous paper [12] we have developed a general theory of magnetic curves in unit tangent sphere bundles. In addition we studied magnetic curves in the unit tangent bundle US^2 of the unit 2-sphere S^2. As a continuation of [12], in this paper, we study magnetic curves in the unit tangent sphere bundle UE^2 of the Euclidean plane E^2. In particular, we obtain all contact normal magnetic curves on UE^2, which satisfy the conservation law. Because the unit tangent sphere bundle UE^2 may be identified as a contact metric manifold with the motion group $E(2)$ of the Euclidean plane E^2, we do some investigations in $E(2)$.

1.1. Magnetic curves

Magnetic curves represent, in physics, the trajectories of charged particles moving on a Riemannian manifold under the action of the magnetic fields. Let (M, g) be a Riemannian manifold and let F be a

closed 2-form on M (often called a *magnetic field* on M). A *magnetic curve* represents a solution of a second order diff erential equation

$$\nabla_{\gamma'}\gamma' = \varphi\gamma', \tag{1.1}$$

where denotes the Levi-Civita connection on M and φ is a skew-symmetric (1, 1) tensor field associated to F, that is $F(X, Y) = g(\varphi X, Y)$ for any vector fields X, Y on M. See e.g. [1]. Such curves are sometimes called also *magnetic geodesics* since the Lorentz equation generalizes the equation of geodesics under arc-length parametrization, namely, $\nabla_{\gamma'}\gamma' = 0$. The equation (1.1) is usually known as the *Lorentz equation*. However, in contrast to the geodesics, magnetic curves cannot be rescaled, because the trajectory of a charged particle depends on the speed γ'. Nevertheless, magnetic curves have constant speed, and hence constant energy, since $\frac{d}{ds} g(\gamma', \gamma') = 2g(\varphi\gamma', \gamma') = 0$.

And now, as usual, we restrict our investigation to a single energy level and we consider only unit speed magnetic curves together with a *strength* $q \in \mathbb{R}$. Therefore, from now on, we study *normal magnetic curve* (i.e. unit speed) satisfying the Lorentz equation

$$\nabla_{\dot\gamma}\dot\gamma = q\,\varphi\dot\gamma, \tag{1.2}$$

where by dot we denote the derivative with respect to the arc-length parameter s.

1.2. Almost contact metric structures

A (ϕ, ξ, η)-*structure* on a manifold M is defined by a field ϕ of endomorphisms of tangent spaces, a vector field ξ and a 1-form η satisfying

$$\eta(\xi) = 1, \quad \phi^2 = -I + \eta \otimes \xi, \quad \phi\xi = 0, \quad \eta \circ \phi = 0.$$

If (M, ϕ, ξ, η) admits a compatible Riemannian metric g, namely

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

for all $X, Y \in \Gamma(TM)$, then M is said to have an *almost contact metric structure*, and (M, ϕ, ξ, η, g) is called an *almost contact metric manifold*. Consequently, we have that ξ is unitary and η is metrically dual to ξ, *i.e.*, $\eta(X) = g(\xi, X)$, for any $X \in \Gamma(TM)$. The vector field ξ is often called the Reeb vector field, even though, this name is used for the contact case.

We define a 2-form Ω on (M, ϕ, ξ, η, g) by

$$\Omega(X, Y) = g(X, \phi Y),$$

for all $X, Y \in \Gamma(TM)$, called the *fundamental 2-form* of the almost contact metric structure (ϕ, ξ, η, g).

The fundamental 2-form is not always closed. However, there are several classes of almost contact metric manifolds with closed fundamental 2-form. For more details see e.g. [3].

Let us remember some more definitions: An almost contact metric manifold is said to be:

(1) A *contact metric manifold* if $\Omega = d\eta$.

(2) An α-*Sasakian manifold* if there exists a constant α such that it satisfies

$$(\nabla_X\phi)Y = \alpha\{g(X, Y)\xi - \eta(Y)X\}$$

for all $X, Y \in \Gamma(TM)$. Recall that a 1-Sasakian manifold is simply called a *Sasakian manifold*, while a 0-Sasakian manifold is called a *cosymplectic manifold*. On an α-Sasakian manifold we have

$$\nabla_X\xi = -\alpha\phi X, \quad d\eta = \alpha\,\Omega.$$

The first formula implies that on α-Sasakian manifolds, ξ is a Killing vector field.

(3) *Normal* if it satisfies

$$[\phi X, \phi Y] + \phi^2[X, Y] - \phi[\phi X, Y] - \phi[X, \phi Y] + 2d\eta(X, Y)\xi = 0$$

for all $X, Y \in \Gamma(TM)$. A normal almost contact metric manifolds is called a *quasi-Sasakian manifold* if Ω is closed.

Note that α-Sasakian manifolds are normal. In particular, Sasakian manifolds are characterized as normal contact metric manifolds.

Let (M, ϕ, ξ, η, g) be an almost contact metric manifold with closed fundamental 2-form Ω. Then for any constant q, the magnetic field $F_q = -q\Omega$ on M is called a *contact magnetic field* with strength q. In our previous papers [7,8,17], we investigated contact magnetic curves in Sasakian manifolds and cosymplectic manifolds, respectively. See also [5,6,9]. In addition, the study of contact magnetic curves in quasi-Sasakian manifolds was initiated in [13,14,16].

2. Magnetic curves in the unit tangent sphere bundle of a Riemannian manifold

In our previous paper [12] we present a detailed study of magnetic curves in the unit tangent sphere bundle $T^{(1)}M$ of a Riemannian manifold (M, g). Recall that the *tangent sphere bundle* of radius $r > 0$ is the hypersurface of the tangent bundle $T(M)$ defined by

$$T^{(r)}M = \{(x; u) \in T(M) : \ g_x(u, u) = r^2\}.$$

On $T^{(r)}M$ we consider the restriction $\bar{g}$ of the Sasaki metric of $T(M)$.

It is well known that the Levi-Civita connection ∇ of (M, g) defines a splitting of the tangent bundle $T(TM)$ of $T(M)$ as follows: $T(TM) = \mathrm{H}\oplus\mathrm{V}$. Here H and V are called the *horizontal* and *vertical* subbundles, respectively. For a vector $X \in T_xM$, the *horizontal lift* of X at a point (x, u) is the unique vector $X^h_u \in \mathrm{H}_u$ such that $\pi_* X^h_u = X_{\pi(u)}$, where $\pi : T(M) \to M$ is the canonical projection. The *vertical lift* of X is the unique vector $X^v_u \in \mathrm{V}_u$ such that $X^v(df) = Xf$, for every C^∞ function on M.

Let $\gamma(s) = (x(s); V(s))$ be a curve in $UM = (T^{(1)}M, \bar{g})$. Then $V(s)$ is a unit vector field along the *base curve* $x(s)$ in M. The curve γ is expressed as

$$\gamma(s) = \left(x^1(s), \dots, x^n(s); V^1(s), \dots, V^n(s)\right).$$

The derivative $\dot{\gamma}(s)$ is written as

$$\dot{\gamma}(s)_{\gamma(s)} = \frac{dx^i}{ds}(s)\frac{\partial}{\partial x^i}\Big|_{\gamma(s)} + \frac{dV^i}{ds}(s)\frac{\partial}{\partial u^i}\Big|_{\gamma(s)}.$$

The following result characterizes contact magnetic curves on the unit tangent sphere bundle of a Riemannian manifold (M, g).

Theorem 2.1 *([12]). A curve $\gamma(s) = (x(s); V(s))$ in UM is a contact magnetic trajectory with strength q if and only if*

$$\begin{aligned} &\nabla_{\dot{x}}\dot{x} + R(V, \nabla_{\dot{x}}V)\dot{x} = -q\,\nabla_{\dot{x}}V, \\ &\nabla_{\dot{x}}\nabla_{\dot{x}}V + g(\nabla_{\dot{x}}V, \nabla_{\dot{x}}V)V - q\dot{x} = -qg(\dot{x}, V)V. \end{aligned} \tag{2.1}$$

Here ∇ is the Levi-Civita connection of g and R is its curvature tensor of type (1, 3).

2.1. Conservation law

We have the following conservation law for geodesics. See e.g. [20].

Proposition 2.1. *Let $\gamma(s) = (x(s); V(s))$ be a geodesic in UM. Then $x(s)$ has constant speed and $\nabla_{\dot{x}}V$ has constant length.*

This result can be extended for magnetic curves as follows.

Proposition 2.2 *([12]). A non-geodesic unit speed contact magnetic curve $\gamma(s) = (x(s); V(s))$ satisfies $|\nabla_{\dot{x}}V| =$ constant (equivalently $|\dot{x}| =$ constant) if and only if $g(\dot{x}, \nabla_{\dot{x}}V) = 0$.*

This conservation law implies that we may reparametrize geodesic $\gamma(s)$ so that $|\dot{x}| = 1$. Under this reparametrization, Sasaki classified geodesic in the unit tangent sphere bundles over space forms.

Theorem 2.2 *([20]). Let M be a Riemannian manifold of constant curvature c. Then the base curve of a geodesic in UM has constant first and second curvatures and vanishing third curvature.*

2.2. Contact angle

Recall that the *contact angle* of a curve γ in an almost contact metric manifold is defined as the angle between its tangent vector field and the Reeb vector field in the corresponding point. For a unit speed curve $\gamma(s)$ in UM we have $\dot{\gamma}(s) = \dot{x}(s)^h_{\gamma(s)} + (\nabla_{\dot{x}}V)^t_{\gamma(s)}$, where by X^t_u we denote the tangential lift of X at a point $u \in UM$. Recall that the tangential lift of a tangent vector X is given by $_uX^t = X^v_u - g_x(X, u)U_u$, where U is the Liouville vector field on $T(M)$. See e.g. [3, Chapter 9] and [4]. On UM, the almost contact metric structure is that induced from the almost Kählerian structure of $T(M)$. The Reeb vector field ξ is precisely the *geodesic spray*, known also as *the geodesic flow* vector field, namely ξ_u is the horizontal vector such that $\pi_{*u}\xi_u = u$. Therefore, the contact angle ϑ of γ is given by

$$\cos\vartheta(s) := \eta_{\gamma(s)}(\dot{\gamma}(s)) = \bar{g}(\dot{\gamma}(s), \xi_{\gamma(s)}) = g_{x(s)}(\dot{x}(s), V(s)),$$

since $\pi_{*\gamma(s)}\xi_{\gamma(s)} = V(s)$. Consequently, a unit speed curve is *slant*, that is the contact angle is constant, if and only if $g(\dot{x}, V)$ is constant.

Motivated by the result of Klingenberg–Sasaki, which says that *any geodesic in* US^2 *is a slant curve in* US^2, we investigate contact magnetic curves which are slant in the unit tangent bundle $UM(c)$ of a space form $M(c)$.

Proposition 2.3 *([12]). Let* $M = M(c)$ *be a space form of curvature* c*. Then any contact normal magnetic curve satisfies the following differential equations system:*

$$\begin{aligned} &\nabla_{\dot{x}}\dot{x} + cg(\dot{x}, \nabla_{\dot{x}}V)V + (q - c\cos\vartheta)\nabla_{\dot{x}}V = 0, \\ &\nabla_{\dot{x}}\nabla_{\dot{x}}V + (|\nabla_{\dot{x}}V|^2 + q\cos\vartheta)V = q\dot{x}. \end{aligned} \qquad \textbf{(MC)}$$

Besides, taking the derivative of $\cos\vartheta(s)$ and using the first equation (**MC**), we get

$$-\sin\vartheta\ \dot{\vartheta} = (1 - c)g(\dot{x}, \nabla_{\dot{x}}V).$$

Accordingly, we obtain the following two results.

Proposition 2.4 *([12]). Every contact normal magnetic curve in* US^n *is slant.*

Proposition 2.5 *([12]). Let* M *be a space form of curvature* $c = 1$*. Then a contact normal magnetic curve is slant if and only if it satisfies the conservation law, that is, both* $|\dot{x}|$ *and* $|\nabla_{\dot{x}}V|$ *are constant.*

3. The unit tangent bundle over Euclidean plane

In this section we will continue our investigation of magnetic curves in the unit tangent bundle of a space form $M(c)$ initiated in [12]. We will consider the case $c = 0$.

3.1. It is well known that the orientation preserving isometry group (*the motion group*) $E(2)$ of the Euclidean plane E^2 may be identified with the unit tangent sphere bundle UE^2.

The rigid motion group of the Euclidean plane E^2 is the semi-direct product of rotation group SO(2) and translation group $(R^2, +)$. The semi-direct product structure of SO(2) R^2 is given by

$$(A, \boldsymbol{p}) \cdot (B, \boldsymbol{q}) := (AB, \boldsymbol{p} + A\boldsymbol{q}), \quad A, B \in SO(2), \ \ \boldsymbol{p}, \boldsymbol{q} \in R^2. \qquad (3.1)$$

It is isomorphic to the following closed subgroup of GL(3, R)

$$E(2) = \left\{ \begin{pmatrix} \cos\psi & -\sin\psi & x \\ \sin\psi & \cos\psi & y \\ 0 & 0 & 1 \end{pmatrix} : x, y \in R, \ \psi \in S^1 \right\}. \qquad (3.2)$$

In fact, we have a global coordinate system of $E(2)$ defined by (x, y, ψ). Thus $E(2)$ is nothing but $R^2(x, y) \rtimes S^1$ with multiplication rule:

$$(x, y, \psi) * (x', y', \psi') = (x + \cos\psi\ x' - \sin\psi\ y', y + \sin\psi\ x' + \cos\psi\ y', \psi + \psi'). \qquad (3.3)$$

The Lie algebra $e(2)$ corresponds to

$$e(2) = \left\{ \begin{pmatrix} 0 & -w & u \\ w & 0 & v \\ 0 & 0 & 0 \end{pmatrix} : u, v, w \in R \right\} \qquad (3.4)$$

and hence we take the following basis $\{E_1, E_2, E_3\}$ of $e(2)$:

Then the left translated vector fields of E_1, E_2, E_3 are

$$e_1 = \cos\psi\frac{\partial}{\partial x} + \sin\psi\frac{\partial}{\partial y}, \quad e_2 = -\sin\psi\frac{\partial}{\partial x} + \cos\psi\frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial \psi},$$

whose Lie brackets satisfy

$$[e_1, e_2] = 0, \quad [e_2, e_3] = e_1, \quad [e_3, e_1] = e_2.$$

We equip $e(2)$ with an inner product $\langle\, ,\, \rangle$ which makes the basis $\{E_1, E_2, E_3\}$ orthonormal. Consider now the left translated Riemannian metric g defined by

$$g = dx^2 + dy^2 + d\psi^2.$$

The dual coframe field $\theta = (\vartheta^1, \vartheta^2, \vartheta^3)$ of $e = (e_1, e_2, e_3)$ is, therefore, given by

$$\vartheta^1 = \cos\psi dx + \sin\psi dy, \quad \vartheta^2 = -\sin\psi dx + \cos\psi dy, \quad \vartheta^3 = d\psi.$$

Finally, the Levi-Civita connection ∇ of $E(2)$ is described by the expressions

$$\nabla_{e_3} e_1 = e_2, \quad \nabla_{e_3} e_2 = -e_1 \text{ and all other } \nabla_{e_i} e_j = 0.$$

3.2. *Canonical contact structure on $E(2)$*

We define a canonical contact structure on the universal covering group $\widetilde{E}(2)$ of $E(2)$.

The universal covering group $\widetilde{E}(2)$ of $E(2)$ is the Cartesian 3-space $\mathbb{R}^3(x, y, \psi)$ endowed with the multiplication:

$$(x, y, \psi) * (x', y', \psi') = (x + \cos\psi\, x' - \sin\psi\, y', y + \sin\psi\, x' + \cos\psi\, y', \psi + \psi'). \qquad (3.5)$$

The discrete group $\Gamma_E = 2\pi\mathbb{Z}$ of $(\mathbb{R}^3, *)$ acts on $\widetilde{E}(2)$ by translation:

$$\widetilde{E}(2) \times \Gamma_E \to \widetilde{E}(2); \quad (x, y, \psi) \cdot 2\pi m = (x, y, \psi + 2\pi m). \qquad (3.6)$$

This action is properly discontinuous. The factor space of $\widetilde{E}(2)$ is diffeomorphic to $\mathbb{R}^2(x, y) \times S^1$ and it is identified with $E(2)$. Let us denote by p_E the projection $p_E : \mathbb{R}^3 \to E(2)$.

On $\widetilde{E}(2) = (\mathbb{R}^3, *)$, the 1-form $\tilde{\eta} = \cos\psi\, dx + \sin\psi\, dy$ is a contact form. Obviously, the contact form $\tilde{\eta}$ induces, via p_E, a contact form η on $E(2)$, that is $p_E^*\eta = \tilde{\eta}$.

We set $\tilde{\xi} = \cos\psi\frac{\partial}{\partial x} + \sin\psi\frac{\partial}{\partial y}$ and we define an endomorphism field ϕ on $\mathbb{R}^3$ by

$$\phi\frac{\partial}{\partial x} = -\sin\psi\frac{\partial}{\partial \psi},$$
$$\phi\frac{\partial}{\partial y} = \cos\psi\frac{\partial}{\partial \psi},$$
$$\phi\frac{\partial}{\partial \psi} = \sin\psi\frac{\partial}{\partial x} - \cos\psi\frac{\partial}{\partial y}.$$

Then $(\phi, \tilde{\xi}, \tilde{\eta})$ is an almost contact structure on $\mathbb{R}^3$. The Euclidean metric $\tilde{g} = dx^2 + dy^2 + d\psi^2$ is related to this almost contact structure by

$$\tilde{g}(\phi X, \phi Y) = \tilde{g}(X, Y) - \tilde{\eta}(X)\tilde{\eta}(Y), \quad 2d\tilde{\eta}(X, Y) = \tilde{g}(X, \phi Y)$$

for all $X, Y \in \mathfrak{X}(\mathbb{R}^3)$.

It is easy to see that this almost contact structure $(\phi, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ induces an almost contact structure on $E(2)$.

The induced almost contact structure (η, ξ, ϕ, g) on $E(2) = (\mathbb{R}^2(x, y) \times S^1, *)$ is given, explicitly, by

$$\eta = \vartheta^1, \ \xi = e_1, \ \phi e_1 = 0, \ \phi e_2 = e_3, \ \phi e_3 = -e_2,$$
$$g = (\vartheta^1)^2 + (\vartheta^2)^2 + (\vartheta^3)^2.$$

3.3. The tangent bundle $T\mathbb{E}^2$ of Euclidean plane $\mathbb{E}^2$ is expressed as

$$T\mathbb{E}^2 = \{(x, y; u, v) \mid x, y, u, v \in \mathbb{R}\} = \mathbb{E}^2 \times \mathbb{E}^2.$$

Then the geodesic spray ξ is given by

$$\xi = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}.$$

The unit tangent sphere bundle $U\mathbb{E}^2$ is parametrized by

$$U\mathbb{E}^2 = \{(x, y; \cos\psi, \sin\psi) \mid x, y \in \mathbb{R},\ \psi \in [0, 2\pi)\} = \mathbb{E}^2 \times S^1.$$

Hence ξ is represented as

$$\xi = \cos\psi\frac{\partial}{\partial x} + \sin\psi\frac{\partial}{\partial y}$$

on $U\mathbb{E}^2$.

With the notations from [12] we have the following quantities:

- the horizontal lifts: $\frac{\partial}{\partial x}^{h} = \frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}^{h} = \frac{\partial}{\partial y}$;
- the vertical lifts: $\frac{\partial}{\partial x}^{v} = \frac{\partial}{\partial u}$, $\frac{\partial}{\partial y}^{v} = \frac{\partial}{\partial v}$;
- the tangential lifts: $\frac{\partial}{\partial x}^{t} = -\sin\psi\frac{\partial}{\partial\psi}$, $\frac{\partial}{\partial y}^{t} = \cos\psi\frac{\partial}{\partial\psi}$.

Therefore, the induced metric (from the Sasaki metric of $T\mathbb{E}^2$) is precisely $g = dx^2 + dy^2 + d\psi^2$. Moreover, the almost contact structure on $U\mathbb{E}^2$ is obtained from the equations

$$\eta(X^h) = \langle X, (\cos\psi, \sin\psi)\rangle, \quad \eta(X^t) = 0,$$
$$\phi X^h = X^t, \qquad \phi X^t = -X^h + \eta(X^h)\xi,$$

for every vector field X on the base manifold $\mathbb{E}^2$. Thus, $U\mathbb{E}^2$ is identified with $E(2)$ as a contact metric manifold.

Take a curve $\gamma : I \longrightarrow U\mathbb{E}^2$, $\gamma(s) = (x(s), y(s), \psi(s))$ parametrized by the arc-length; then we write

$$\begin{aligned}\dot\gamma(s) &= \dot x(s)\partial_x + \dot y(s)\partial_y + \dot\psi(s)\partial_\psi\\ &= \dot x(s)(\cos\psi(s)e_1 - \sin\psi(s)e_2) + \dot y(s)(\sin\psi(s)e_1 + \cos\psi(s)e_2) + \dot\psi(s)e_3\\ &= (\dot x(s)\cos\psi(s) + \dot y(s)\sin\psi(s))e_1 + (-\dot x(s)\sin\psi(s) + \dot y(s)\cos\psi(s))e_2 + \dot\psi(s)e_3.\end{aligned}$$

We compute

$$\begin{aligned}\nabla_{\dot\gamma}\dot\gamma(s) = &\tfrac{d}{ds}(\dot x(s)\cos\psi(s) + \dot y(s)\sin\psi(s))e_1 + \tfrac{d}{ds}(-\dot x(s)\sin\psi(s) + \dot y(s)\cos\psi(s))e_2 + \ddot\psi(s)e_3 +\\ &+ (\dot x(s)\cos\psi(s) + \dot y(s)\sin\psi(s))\dot\psi(s)e_2 - (-\dot x(s)\sin\psi(s) + \dot y(s)\cos\psi(s))\dot\psi(s)e_1.\end{aligned}$$

Moreover, we have

$$\phi\dot\gamma(s) = \left(-\dot x(s)\sin\psi(s) + \dot y(s)\cos\psi(s)\right) e_3 - \dot\psi(s)e_2.$$

The Lorentz equation becomes

$$\begin{aligned}&(\dot x(s)\cos\psi(s) + \dot y(s)\sin\psi(s))^{\cdot} - (-\dot x(s)\sin\psi(s) + \dot y(s)\cos\psi(s))\dot\psi(s) = 0,\\ &(-\dot x(s)\sin\psi(s) + \dot y(s)\cos\psi(s))^{\cdot} + (\dot x(s)\cos\psi(s) + \dot y(s)\sin\psi(s))\dot\psi(s) = -q\dot\psi(s),\\ &\ddot\psi(s) = q(-\dot x(s)\sin\psi(s) + \dot y(s)\cos\psi(s)).\end{aligned}$$

We immediately obtain the following system of ordinary diff erential equations

$$\cos\psi(s)\ddot x(s) + \sin\psi(s)\ddot y(s) = 0,$$

$$-\sin \Box(s)\ddot{x}(s) + \cos \Box(s)\ddot{y}(s) = -q\Box(s),$$
$$\Box(s) = q(-\dot{x}(s)\sin \Box(s) + \dot{y}(s)\cos \Box(s)),$$

which is equivalent to

$$\begin{aligned} \ddot{x}(s) &= q \sin \Box(s)\,\Box(s), \\ \ddot{y}(s) &= -q \cos \Box(s)\,\Box(s), \\ \Box(s) &= q(-\dot{x}(s)\sin \Box(s) + \dot{y}(s)\cos \Box(s)). \end{aligned} \tag{3.7}$$

Note that, the curve γ is a geodesic if and only if

$$\ddot{x}(s) = \ddot{y}(s) = \Box(s) = 0.$$

The first and second equations of (3.7) can be rewritten as

$$\frac{d}{ds}(\dot{x}(s) + q\cos \Box(s)) = \frac{d}{ds}(\dot{x}(s) + q\sin \Box(s)) = 0.$$

Hence there exist constants c_1 and c_2 such that

$$\dot{x}(s) = c_1 - q\cos \Box(s), \quad \dot{y}(s) = c_2 - q\sin \Box(s). \tag{3.8}$$

Inserting these equations into the third equation of (3.7), we get

$$\Box(s) = q(-c_1 \sin \Box(s) + c_2 \cos \Box(s)).$$

When $\Box$ is nonconstant, multiplying this diff erential equation by $\Box(s)$, we get

$$\Box(s)\Box(s) = q(-c_1 \sin \Box(s) + c_2 \cos \Box(s))\Box(s),$$

which leads to

$$\Box(s)^2 = 2q\ c_1 \cos \Box(s) + c_2 \sin \Box(s)\ + c_3, \quad c_3 \in \mathrm{R}. \tag{3.9}$$

Hence, the equations of motion for γ are built in the following theorem.

Theorem 3.1. *Let $\gamma : I \longrightarrow \widetilde{U\mathrm{E}^2}$, $\gamma(s) = (x(s), y(s), \Box(s))$, be a normal contact magnetic curve with strength q. Then its velocity satisfies (3.8) and (3.9), where c_1, c_2 and c_3 are real constants such that $c_1^2 + c_2^2 + q^2 + c_3 = 1$.*

The ordinary diff erential equation (3.9) can be solved in terms of elliptic functions and/or hypergeometric functions depending on the constants c_1 and c_2. In this paper, we do not intend to give all explicit solutions (as, for example, in [18]); so, from now on, we consider only slant contact normal magnetic curves. By Proposition 2.5 we know that γ satisfies the conservation law, meaning that $\dot{x}^2(s) + \dot{y}^2(s)$ is constant and hence $\Box(s)$ is constant, too. As $\Box$ is not a constant (since γ is not a geodesic) and $q = 0$, it follows, from (3.9), that $c_1 = c_2 = 0$, which implies $\Box(s)^2 = c_3 = 1 - q^2$.

We state the following theorem.

Theorem 3.2. *Let $\gamma : I \longrightarrow \widetilde{U\mathrm{E}^2}$, $\gamma(s) = (x(s), y(s), \Box(s))$, be a slant, non-geodesic, normal contact magnetic curve with strength q. Then γ is given by*

$$x(s) = -\frac{q}{\sqrt{1-q^2}}\sin(s\sqrt{1-q^2}), \quad y(s) = \frac{q}{\sqrt{1-q^2}}\cos(s\sqrt{1-q^2}), \quad \Box(s) = s\sqrt{1-q^2}.$$

Here we have set the initial point at $\gamma(0) = \left(0, q/\sqrt{1-q^2}, 0\right)$. Moreover, $|q| \in (0, 1)$.

Looking back to the unit tangent bundle UE^2 we remark that the base curve is a circle, centered at the origin and with radius $\frac{|q|}{\sqrt{1-q^2}}$ and the unit tangent vector along this circle is, up to orientation, the normalized tangent vector, that is $V(s) = \left(\cos(s\sqrt{1-q^2}), \sin(s\sqrt{1-q^2})\right)$.

Finally, we remark that in case when [illegible] is a constant, we obtain that the base curve is a straight line in E^2, while V is any constant unit vector field along this line.

4. Magnetic curves on tori

4.1. *Helicoidal motion*

Let $E^3(x, y, z)$ be the Euclidean 3-space with natural Riemannian metric $dx^2 + dy^2 + dz^2$ as before. For any vector $\boldsymbol{v} \in R^3$, we denote by $\Gamma(\boldsymbol{v})$ the discrete subgroup of $(R^3, +)$ generated by $\boldsymbol{v}$:

$$\Gamma(\boldsymbol{v}) = \{ 2\pi m\boldsymbol{v} \mid m \in Z \}.$$

In the special case when $\boldsymbol{v} = \boldsymbol{e}_3 = (0, 0, 1)$, the factor space $E^3/\Gamma(\boldsymbol{e}_3)$ is precisely the underlying flat Riemannian manifold of $E(2)$. This fundamental observation inspired some researchers to use group structure of $E(2)$ for the study of several geometric objects in Euclidean 3-space which are invariant under the $\Gamma(\boldsymbol{e}_3)$-action. See e.g. [10].

For instance, the left translation by $(\alpha, \beta, \gamma) \in \widetilde{E}(2)$ can be rewritten as

$$L_{(\alpha,\beta,\gamma)}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}.$$

In particular, the one parameter subgroup $\{(0, 0, \gamma) \mid \gamma \in R\} \subset \widetilde{E}(2)$ acts on R^3 as *helicoidal motion group* (or *screw motion group*) of pitch 1:

$$L_{(0,0,\gamma)}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \gamma \end{bmatrix}.$$

The action of $\{(0, 0, \gamma) \mid \gamma \in R\}$ decends to an $SO(2)$-action on $E(2)$.

For example, the helicoid $z = \tan^{-1}(y/x)$ is an orbit of the x-axis under the action of the one-parameter subgroup $\{(0, 0, v) \mid v \in R\} \cong (R(v), +)$. It is given by the immersion

$$\tilde{f} : R^2(u, v) \to E^3, \quad \tilde{f}(u, v) = (u\cos v,\ u\sin v,\ v) = L_{(0,0,v)}(u, 0, 0).$$

The helicoid $\tilde{f}$ induces a minimal surface f in $E^3/\Gamma(\boldsymbol{e}_3)$. In $E^3/\Gamma(\boldsymbol{e}_3)$, the helicoid f is regarded as S^1-orbit of the x-axis. Namely, with respect to the group structure $*$, helicoid is regarded as a "rotational surface" in $E^3/\Gamma(\boldsymbol{e}_3)$.

Finally, observe that $(0, 0, v) * (u, 0, 0) \neq (u, 0, 0) * (0, 0, v) = (u, 0, v)$. The image of this immersion is xz-plane. Obviously, this is also $\Gamma(\boldsymbol{e}_3)$-invariant.

4.2. *Standard contact structure on tori*

The canonical contact structure on $E(2)$ induces a contact structure on 3-dimensional tori. In the following, we briefly exhibit the induced contact structure on flat tori.

The contact form $\tilde{\eta}$ is invariant under the action of discrete subgroup $\Gamma_T = 2\pi Z^3$ of $(R^3, +)$ defined by

$$(x, y, z) + 2\pi(l, m, n),\ (l, m, n) \in Z^3.$$

Furthermore the Euclidean metric $\tilde{g} = dx^2 + dy^2 + dz^2$ is invariant under Γ_T, too. Hence $\tilde{\eta}$ induces a contact structure η_T on the (flat) torus $T^3 = R^3/\Gamma_T$. Thus (T^3, η_T) is a compact flat 3-manifold which admits a contact structure. We give the following result.

Proposition 4.1. *The 3-torus $T^3 = \widetilde{E}(2)/2\pi Z^3$ admits a contact structure.*

Note that on Γ_T, two multiplications "+" and "$*$" coincide. Hence the factor space $(R^3, *)/2\pi Z^3$ is a 3-torus with "noncommutative" Lie group structure.

Besides, this contact structure on T^3 is not regular. See also [2]. The Reeb vector field $\tilde{\xi}$ of $(R^3, \tilde{\eta})$ is

$$\tilde{\xi} = \cos z\frac{\partial}{\partial x} + \sin z\frac{\partial}{\partial y}.$$

The integral curve $\psi(t)$ of $\tilde{\xi}$ through $(0, 0, \frac{\pi}{3})$ is $\psi(t) = \left(\frac{t}{2}, \frac{\sqrt{3}t}{2}, \frac{\pi}{3}\right)$. Hence ξ induces an irrational flow on 2-torus in T^3 defined by $z = \frac{\pi}{3}$. Thus the 3-torus T^3 is not a regular contact manifold. In fact, every 3-torus cannot admit regular contact structure. More generally, Blair proved that no torus T^{2n+1} can carry a regular contact structure. See e.g. [3, p. 52].

The almost contact structure $(\phi, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ on $\tilde{E}(2)$ induces an almost contact structure on T^3. However, in order to connect our work to contact metric geometry, we have to renormalize some objects, that is we consider

$$\hat{\eta} := \frac{1}{2}\tilde{\eta}, \ \hat{\xi} := 2\tilde{\xi}, \ \phi := \phi, \ \hat{g} := \frac{1}{4}\tilde{g}.$$

Then $\hat{M}^3 = (\mathbb{R}^3, \hat{\eta}, \hat{\xi}, \phi, \hat{g})$ is a contact metric manifold, and hence T^3 together with contact metric structure induced from $\hat{M}^3$ provides an example of compact contact metric manifold which is not regular.

4.3. *Conclusions and further research*

The contact metric structure on $\mathbb{R}^3$ and T^3 are *not* homogeneous with respect to the additive group structure but homogeneous with respect to the group structure of $E(2)$. In particular, $E(2)$ itself is a homogeneous contact metric manifold.

According to this observation, it seems to be natural that the contact structure determined by $\tilde{\eta}$ on $\mathbb{R}^3$ is regarded as a contact structure on the covering group $\tilde{E}(2)$ of the Euclidean motion group (from the group theoretical viewpoints).

In [18] the authors consider two magnetic fields on the 3-torus obtained from two different contact forms on the Euclidean 3-space. They study when their corresponding normal magnetic curves are closed obtaining
some periodicity conditions involving the set of rational numbers. Remark that the left invariant contact structure on $\tilde{E}(2)$ compatible to the flat metric induces a contact structure on the 3-torus T^3 which coincides with the non-regular one used in [18].

These observations could motivate researchers to study contact magnetic curves on homogeneous contact metric 3-manifolds. As is well known, homogeneous contact metric 3-manifolds are realized as 3-dimensional Lie groups equipped with left invariant contact metric structure (see [11,19]).

To conclude, we would like to emphasize the paper [15], where the authors study the algebra of the integral motion of magnetic geodesic flows and their integrability in 4-dimensional Lie groups.

As further research we propose the study of contact magnetic curves in 3-dimensional Lie groups equipped with left invariant contact metric structures.

Acknowledgments

The first author was partially supported by JSPS KAKENHI Grant no. 15K04834. The second author was supported by a mobility grant of the Romanian Ministery of Research and Innovation, CNCS-UEFISCDI, project number PN-III-P1-1.1-MC-2018-0251, within PNCDI III. The authors wish to thank the referee for carefully reading the manuscript and for giving constructive suggestions which substantially helped to increase the quality of the paper.

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