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Location of Zeros of Polynomials

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ABSTRACT
In this paper, restricting the coefficients of a polynomial to certain conditions, we locate a region containing all of its zeros. Our results generalize many known results in addition to some interesting results which can be obtained by choosing certain values of the parameters.
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I. INTRODUCTION

On the location of zeros of a polynomial with real coefficients Enestrom and Kakeya proved the following theorem named after them as the Enestrom - Kakeya Theorem [9,10] :

Theorem A: all the zeros of a polynomial $P(z)=\sum_{j=0}^{n} a_j z^j$ satisfying $a_n \ge a_{n-1} \ge ...... \ge a_1 \ge a_0 > 0$ lie in $|z| \le 1$.

The above theorem has been generalized in different ways by various authors [2-11].
Recently Gulzar et al [7] proved the following result

Theorem B: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $\mathrm{Re}(a_j)=\alpha_j, \mathrm{Im}(a_j)=\beta_j, j=0,1,2,......, n$ such that for some $\lambda, \mu, 0 \le \lambda \le n-1, 0 \le \mu \le n-1$,

$$\alpha_n \ge \alpha_{n-1} \ge ...... \ge \alpha_\lambda,$$

$$\beta_n \ge \beta_{n-1} \ge ...... \ge \beta_\mu,$$

and

$$L = |\alpha_\lambda - \alpha_{\lambda-1}| + |\alpha_{\lambda-1} - \alpha_{\lambda-2}| + ... + |\alpha_1 - \alpha_0| + |\alpha_0|,$$

$$M = |\beta_\mu - \beta_{\mu-1}| + |\beta_{\mu-1} - \beta_{\mu-2}| + ... + |\beta_1 - \beta_0| + |\beta_0|.$$

Then all the zeros of P(z) lie in

$$|z| \le \frac{1}{|a_n|}[\alpha_n + \beta_n + L + M - \alpha_\lambda - \beta_\mu].$$

II. MAIN RESULTS

In this paper we prove the following generalization of Theorem B:

Theorem 1: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $\mathrm{Re}(a_j)=\alpha_j, \mathrm{Im}(a_j)=\beta_j, j=0,1,2,......, n$ such that for some $\lambda, \mu, 0 \le \lambda \le n-1, 0 \le \mu \le n-1$ and for some $k_1, k_2 \ge 1$,

$$k_1^{n-\lambda+1}\alpha_n \ge k_1^{n-\lambda}\alpha_{n-1} \ge ...... \ge k_1^2\alpha_{\lambda+1} \ge k_1\alpha_\lambda,$$

$$k_2^{n-\mu+1}\beta_n \ge k_2^{n-\mu}\beta_{n-1} \ge ...... \ge k_2^2\beta_\mu \ge k_2\beta_\mu,$$

and

$$L = |\alpha_\lambda - \alpha_{\lambda-1}| + |\alpha_{\lambda-1} - \alpha_{\lambda-2}| + ... + |\alpha_1 - \alpha_0| + |\alpha_0|,$$

$$M = |\beta_\mu - \beta_{\mu-1}| + |\beta_{\mu-1} - \beta_{\mu-2}| + ... + |\beta_1 - \beta_0| + |\beta_0|.$$

Then all the zeros of P(z) lie in

$$\left| z + \frac{(k_1 - 1)\alpha_n + i(k_2 - 1)\beta_n}{a_n} \right| \le \frac{1}{|a_n|}[\alpha_n + \beta_n + (k_1 - 1)\sum_{j=\lambda+1}^{n} (|\alpha_j| + \alpha_j) - (k_2 - 1)\sum (|\beta_j| + \beta_j)$$

$$- (k_1 - 1)|\alpha_n| - (k_2 - 1)|\beta_n| + L + M - \alpha_\lambda - \beta_\mu].$$

Further the number of zeros of P(z) in $\frac{|a_0|}{X} \le |z| \le \frac{R}{c}, c > 1, R \ge 1$ does not exceed

$\frac{1}{\log c} \log \frac{X + |a_0|}{|a_0|} = \frac{1}{\log c} \log(1 + \frac{X}{|a_0|})$ and the number of zeros of P(z) in $\frac{|a_0|}{Y} \le |z| \le \frac{R}{c}, c > 1, R \le 1$

does not exceed $\frac{1}{\log c} \log \frac{Y + |a_0|}{|a_0|} = \frac{1}{\log c} \log(1 + \frac{Y}{|a_0|})$, where R is any positive number and

$$X = |a_n| R^{n+1} + R^n [(\alpha_n + \beta_n) - (\alpha_\lambda + \beta_\mu) + L + M - |\alpha_0| - |\beta_0| + (k_1 - 1)\sum_{j=\lambda+1}^{n} (|\alpha_j| + \alpha_j)$$

$$+ (k_2 - 1)\sum_{j=\mu+1}^{n} (|\beta_j| + \beta_j)],$$

$$Y = |a_n| R^{n+1} + R[(\alpha_n + \beta_n) - (\alpha_\lambda + \beta_\mu) + L + M - |\alpha_0| - |\beta_0| + (k_1 - 1)\sum_{j=\lambda+1}^{n} (|\alpha_j| + \alpha_j)$$

$$+ (k_2 - 1)\sum_{j=\mu+1}^{n} (|\beta_j| + \beta_j)].$$

Remark 1: For $k_1 = k_2 = 1$, Theorem 1 reduces to Theorem B.

Taking $k_2 = 1$ in Theorem 1, we get the following result:

Corollary 1: Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $\mathrm{Re}(a_j) = \alpha_j, \mathrm{Im}(a_j) = \beta_j, j = 0,1,2,......, n$ such that for some $\lambda, \mu, 0 \le \lambda \le n-1, 0 \le \mu \le n-1$ and for some $k_1 \ge 1$,

$$k_1^{n-\lambda+1}\alpha_n \ge k_1^{n-\lambda}\alpha_{n-1} \ge ...... \ge k_1^2\alpha_{\lambda+1} \ge k_1\alpha_\lambda,$$

$$\beta_n \ge \beta_{n-1} \ge ...... \ge \beta_\mu \ge \beta_\mu,$$

$$L = |\alpha_\lambda - \alpha_{\lambda-1}| + |\alpha_{\lambda-1} - \alpha_{\lambda-2}| + ... + |\alpha_1 - \alpha_0| + |\alpha_0|,$$

$$M = |\beta_\mu - \beta_{\mu-1}| + |\beta_{\mu-1} - \beta_{\mu-2}| + ... + |\beta_1 - \beta_0| + |\beta_0|.$$

Then all the zeros of P(z) lie in

$$|z + (k_1 - 1)\alpha_n| \le \frac{1}{|a_n|}[\alpha_n + \beta_n + (k_1 - 1)\sum_{j=\lambda+1}^{n} (|\alpha_j| + \alpha_j) - (k_1 - 1)|\alpha_n|$$

$$+ L + M - \alpha_\lambda - \beta_\mu].$$

Also the number of zeros of P(z) in $\frac{|a_0|}{X} \le |z| \le \frac{R}{c}, c > 1, R \ge 1$ does not exceed

$\frac{1}{\log c} \log \frac{X + |a_0|}{|a_0|} = \frac{1}{\log c} \log(1 + \frac{X}{|a_0|})$ and the number of zeros of P(z) in $\frac{|a_0|}{Y} \le |z| \le \frac{R}{c}, c > 1, R \le 1$

does not exceed $\frac{1}{\log c} \log \frac{Y + |a_0|}{|a_0|} = \frac{1}{\log c} \log(1 + \frac{Y}{|a_0|})$, where R is any positive number and

$$X = |a_n| R^{n+1} + R^n [(\alpha_n + \beta_n) - (\alpha_\lambda + \beta_\mu) + L + M - |\alpha_0| - |\beta_0| + (k_1 - 1) \sum_{j=\lambda+1}^{n} (|\alpha_j| + \alpha_j)],$$

$$Y = |a_n| R^{n+1} + R[(\alpha_n + \beta_n) - (\alpha_\lambda + \beta_\mu) + L + M - |\alpha_0| - |\beta_0| + (k_1 - 1) \sum_{j=\lambda+1}^{n} (|\alpha_j| + \alpha_j)]$$

Taking $\lambda = \mu = 0$ in Theorem 1, we get the following result:

Corollary 2: Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $\text{Re}(a_j) = \alpha_j, \text{Im}(a_j) = \beta_j, j = 0,1,2,......, n$ such that for some $\lambda, \mu, 0 \le \lambda \le n-1, 0 \le \mu \le n-1$ and for some $k_1, k_2 \ge 1$,

$$k_1^n \alpha_n \ge k_1^{n-1} \alpha_{n-1} \ge ...... \ge k_1 \alpha_1 \ge \alpha_0,$$

$$k_2^n \beta_n \ge k_2^{n-1} \beta_{n-1} \ge ...... \ge k_2 \beta_1 \ge \beta_0,$$

Then all the zeros of P(z) lie in

$$\left| z + \frac{(k_1 - 1)\alpha_n + i(k_2 - 1)\beta_n}{a_n} \right| \le \frac{1}{|a_n|} [\alpha_n + \beta_n + (k_1 - 1) \sum_{j=1}^{n} (|\alpha_j| + \alpha_j) + (k_2 - 1) \sum_{j=1}^{n} (|\beta_j| + \beta_j)$$

$$- (k_1 - 1)|\alpha_n| - (k_2 - 1)|\beta_n| + |\alpha_0| + |\beta_0| - \alpha_0 - \beta_0].$$

Also the number of zeros of P(z) in $\frac{|a_0|}{X} \le |z| \le \frac{R}{c}, c > 1, R \ge 1$ does not exceed

$\frac{1}{\log c} \log \frac{X + |a_0|}{|a_0|} = \frac{1}{\log c} \log(1 + \frac{X}{|a_0|})$ and the number of zeros of P(z) in $\frac{|a_0|}{Y} \le |z| \le \frac{R}{c}, c > 1, R \le 1$

does not exceed $\frac{1}{\log c} \log \frac{Y + |a_0|}{|a_0|} = \frac{1}{\log c} \log(1 + \frac{Y}{|a_0|})$, where R is any positive number and

$$X = |a_n| R^{n+1} + R^n [(\alpha_n + \beta_n) - (\alpha_0 + \beta_0) + L + M - |\alpha_0| - |\beta_0| + (k_1 - 1) \sum_{j=1}^{n} (|\alpha_j| + \alpha_j)],$$

$$+ (k_2 - 1) \sum_{j=1}^{n} (|\beta_j| + \beta_j)] \ , \ R \ge 1$$

$$Y = |a_n| R^{n+1} + R[(\alpha_n + \beta_n) - (\alpha_0 + \beta_0) + L + M - |\alpha_0| - |\beta_0| + (k_1 - 1) \sum_{j=1}^{n} (|\alpha_j| + \alpha_j)]$$

$$+ (k_2 - 1) \sum_{j=1}^{n} (|\beta_j| + \beta_j)] \ , \ R \le 1$$

Similarly for other values of the parameters in Theorem 1, we get many other interesting results.

III. LEMMAS

For the proof of Theorem 2, we make use of the following lemmas:

Lemma 1: Let f(z) (not identically zero) be analytic for $|z| \le R$, $f(0) \ne 0$ and $f(a_k) = 0$, $k = 1,2,......, n$. Then

$$\frac{1}{2\pi}\int_0^{2\pi} \log \left| f(Re^{i\theta}\right| d\theta - \log |f(0)| = \sum_{j=1}^{n} \log \frac{R}{|a_j|}.$$

Lemma 1 is the famous Jensen's Theorem (see page 208 of [1]).

Lemma 2: Let f(z) be analytic, $f(0) \ne 0$ and $|f(z)| \le M$ for $|z| \le R$. Then the number of zeros of f(z) in $|z| \le \frac{R}{c}, c > 1$ is less than or equal to $\frac{1}{\log c} \log \frac{M}{|f(0)|}$.

Lemma 2 is a simple deduction from Lemma 1.

IV. PROOF OF THEOREM 1

Consider the polynomial

$$\begin{aligned}
F(z) &= (1-z)P(z) \\
&= (1-z)(a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0) \\
&= -a_n z^{n+1} + (a_n - a_{n-1}) z^n + ... + (a_{\lambda+1} - a_\lambda) z^{\lambda+1} + (a_\lambda - a_{\lambda-1}) z^\lambda \\
&\quad + ... + (a_1 - a_0) z + a_0 \\
&= -a_n z^{n+1} - (k_1 - 1)\alpha_n z^n + (k_1\alpha_n - \alpha_{n-1}) z^n + (k_1 \alpha_{n-1} - \alpha_{n-2}) z^{n-1} + ... \\
&\quad + (k_1 \alpha_{\lambda+1} - \alpha_\lambda) z^{\lambda+1} + (\alpha_\lambda - \alpha_{\lambda-1}) z^\lambda + ... + (\alpha_1 - \alpha_0) z + \alpha_0 \\
&\quad - (k_1 - 1)(\alpha_{n-1} z^{n-1} + \alpha_{n-2} z^{n-2} + ... + \alpha_{\lambda+1} z^{\lambda+1}) \\
&\quad + i\{(k_2 \beta_n - \beta_{n-1}) z^n - (k_2 - 1)\beta_n z^n + (k_2 \beta_{n-1} - \beta_{n-2}) z^{n-1} + ... \\
&\quad + (k_2 \beta_{\mu+1} - \beta_\mu) z^{\mu+1} - (k_2 - 1)(\beta_{n-1} z^{n-1} + ... + \beta_{\mu+1} z^{\mu+1}) + (\beta_\mu - \beta_{\mu-1}) z^\mu \\
&\quad + ... + (\beta_1 - \beta_0) z + \beta_0\}
\end{aligned}$$

For $|z| > 1$ so that $\frac{1}{|z|^j} < 1, \forall j = 1,2,......, n$, we have, by using the hypothesis

$$\begin{aligned}
|F(z)| &\ge |a_n z + (k_1 - 1)\alpha_n + i(k_2 - 1)\beta_n||z|^n - [|k_1\alpha_n - \alpha_{n-1}||z|^n + |k_1 \alpha_{n-1} - \alpha_{n-2}||z|^{n-1} + ... \\
&\quad + |k_1 \alpha_{\lambda+1} - \alpha_\lambda||z|^{\lambda+1} + |\alpha_\lambda - \alpha_{\lambda-1}||z|^\lambda + ... + |\alpha_1 - \alpha_0||z| + |\alpha_0| \\
&\quad + (k_1 - 1)(|\alpha_{n-1}||z|^{n-1} + ... + |\alpha_{\lambda+1}||z|^{\lambda+1}) + |k_2 \beta_n - \beta_{n-1}||z|^n + |k_2 \beta_{n-1} - \beta_{n-2}||z|^{n-1} \\
&\quad + ... + |k_2 \beta_{\mu+1} - \beta_\mu||z|^{\mu+1} + |\beta_\mu - \beta_{\mu-1}||z|^\mu + ... + |\beta_1 - \beta_0||z| + |\beta_0| \\
&\quad + (k_2 - 1)(|\beta_{n-1}||z|^{n-1} + ... + |\beta_{\mu+1}||z|^{\mu+1})] \\
&= |z|^n [|a_n z + (k_1 - 1)\alpha_n + i(k_2 - 1)\beta_n| - [|k_1\alpha_n - \alpha_{n-1}| + \frac{|k_1\alpha_{n-1} - \alpha_{n-2}|}{|z|} + \frac{|k_1\alpha_{n-2} - \alpha_{n-3}|}{|z|^2}
\end{aligned}$$

$$+ \ldots\ldots + \frac{|k_1\alpha_{\lambda+1} - \alpha_\lambda|}{|z|^{n-\lambda-1}} + \frac{|\alpha_\lambda - \alpha_{\lambda-1}|}{|z|^{n-\lambda}} + \ldots\ldots + \frac{|\alpha_1 - \alpha_0|}{|z|^{n-1}} + \frac{|\alpha_0|}{|z|^n} + (k_1 - 1)(\frac{|\alpha_{n-1}|}{|z|} + \ldots\ldots$$

$$+ \frac{|\alpha_{\lambda+1}|}{|z|^{n-\lambda-1}}) + |k_2\beta_n - \beta_{n-1}| + \frac{|k_2\beta_{n-1} - \beta_{n-2}|}{|z|} + \ldots\ldots + \frac{|k_2\beta_{\mu+1} - \beta_\mu|}{|z|^{n-\mu-1}} + \frac{|\beta_\mu - \beta_{\mu-1}|}{|z|^{n-\mu}}$$

$$+ \ldots\ldots + \frac{|\beta_1 - \beta_0|}{|z|^{n-1}} + \frac{|\beta_0|}{|z|^n} + (k_2 - 1)(\frac{|\beta_{n-1}|}{|z|} + \ldots\ldots + \frac{|\beta_{\mu+1}|}{|z|^{n-\mu-1}})]$$

$$> |z|^n [|a_n z + (k_1 - 1)\alpha_n + i(k_2 - 1)\beta_n| - [|k_1\alpha_n - \alpha_{n-1}| + |k_1\alpha_{n-1} - \alpha_{n-2}| + |k_1\alpha_{n-2} - \alpha_{n-3}|$$

$$+ \ldots\ldots + |k_1\alpha_{\lambda+1} - \alpha_\lambda| + |\alpha_\lambda - \alpha_{\lambda-1}| + \ldots\ldots + |\alpha_1 - \alpha_0| + |\alpha_0| + (k_1 - 1)(|\alpha_{n-1}| + \ldots\ldots$$

$$+ |\alpha_{\lambda+1}|) + |k_2\beta_n - \beta_{n-1}| + |k_2\beta_{n-1} - \beta_{n-2}| + \ldots\ldots + |k_2\beta_{\mu+1} - \beta_\mu| + |\beta_\mu - \beta_{\mu-1}|$$

$$+ \ldots\ldots + |\beta_1 - \beta_0| + |\beta_0| + (k_2 - 1)(|\beta_{n-1}| + \ldots\ldots + |\beta_{\mu+1}|)]]$$

$$= |z|^n [|a_n z + (k_1 - 1)\alpha_n + i(k_2 - 1)\beta_n| - \{k_1\alpha_n - \alpha_{n-1} + k_1\alpha_{n-1} - \alpha_{n-2} + k_1\alpha_{n-2} - \alpha_{n-3}$$

$$+ \ldots\ldots + k_1\alpha_{\lambda+1} - \alpha_\lambda + |\alpha_\lambda - \alpha_{\lambda-1}| + \ldots\ldots + |\alpha_1 - \alpha_0| + |\alpha_0| + (k_1 - 1)(|\alpha_{n-1}| + \ldots..$$

$$+ |\alpha_{\lambda+1}|) + k_2\beta_n - \beta_{n-1} + k_2\beta_{n-1} - \beta_{n-2} + \ldots\ldots + k_2\beta_{\mu+1} - \beta_\mu + |\beta_\mu - \beta_{\mu-1}|$$

$$+ \ldots\ldots + |\beta_1 - \beta_0| + |\beta_0| + (k_2 - 1)(|\beta_{n-1}| + \ldots\ldots + |\beta_{\mu+1}|)\}]$$

$$= |z|^n [|a_n z + (k_1 - 1)\alpha_n + i(k_2 - 1)\beta_n| - \{k_1\alpha_n - \alpha_{n-1} + k_1\alpha_{n-1} - \alpha_{n-2} + k_1\alpha_{n-2} - \alpha_{n-3}$$

$$+ \ldots\ldots + k_1\alpha_{\lambda+1} - \alpha_\lambda + L + (k_1 - 1)(|\alpha_{n-1}| + \ldots\ldots + |\alpha_{\lambda+1}|) + k_2\beta_n - \beta_{n-1}$$

$$+ k_2\beta_{n-1} - \beta_{n-2} + \ldots\ldots + k_2\beta_{\mu+1} - \beta_\mu + M + (k_2 - 1)(|\beta_{n-1}| + \ldots\ldots + |\beta_{\mu+1}|)\}]$$

$$= |z|^n [|a_n z + (k_1 - 1)\alpha_n + i(k_2 - 1)\beta_n| - \{\alpha_n + \beta_n + (k_1 - 1)\sum_{j=\lambda+1}^{n}(|\alpha_j| + \alpha_j)$$

$$+ (k_2 - 1)\sum_{j=\mu+1}^{n}(|\beta_j| + \beta_j) - (k_1 - 1)|\alpha_n| - (k_2 - 1)|\beta_n| + L + M - \alpha_\lambda - \beta_\mu\}]$$

$$> 0$$

if

$$|a_n z + (k_1 - 1)\alpha_n + i(k_2 - 1)\beta_n| > \{\alpha_n + \beta_n + (k_1 - 1)\sum_{j=\lambda+1}^{n}(|\alpha_j| + \alpha_j) + (k_2 - 1)\sum_{j=\mu+1}^{n}(|\beta_j| + \beta_j)$$
$$- (k_1 - 1)|\alpha_n| - (k_2 - 1)|\beta_n| + L + M - \alpha_\lambda - \beta_\mu\}]$$

i.e.if

$$\left|z + \frac{(k_1 - 1)\alpha_n + i(k_2 - 1)\beta_n}{a_n}\right| > \frac{1}{|a_n|}[\alpha_n + \beta_n + (k_1 - 1)\sum_{j=\lambda+1}^{n}(|\alpha_j| + \alpha_j) + (k_2 - 1)\sum_{j=\mu+1}^{n}(|\beta_j| + \beta_j)$$
$$- (k_1 - 1)|\alpha_n| - (k_2 - 1)|\beta_n| + L + M - \alpha_\lambda - \beta_\mu].$$

This shows that those zeros of F(z) whose modulus is greater than 1 lie in

$$\left|z + \frac{(k_1 - 1)\alpha_n + i(k_2 - 1)\beta_n}{a_n}\right| \le \frac{1}{|a_n|}[\alpha_n + \beta_n + (k_1 - 1)\sum_{j=\lambda+1}^{n}(|\alpha_j| + \alpha_j) + (k_2 - 1)\sum_{j=\mu+1}^{n}(|\beta_j| + \beta_j)$$
$$- (k_1 - 1)|\alpha_n| - (k_2 - 1)|\beta_n| + L + M - \alpha_\lambda - \beta_\mu].$$

Since the zeros of F(z) whose modulus is less than or equal to 1 already satisfy the above inequality and since the zeros of P(z) are also the zeros of F(z) , it follows that all the zeros of P(z) lie in

$$\left|z+\frac{(k_1-1)\alpha_n+i(k_2-1)\beta_n}{a_n}\right|\le\frac{1}{|a_n|}[\alpha_n+\beta_n+(k_1-1)\sum_{j=\lambda+1}^{n}(|\alpha_j|+\alpha_j)+(k_2-1)\sum_{j=\mu+1}^{n}(|\beta_j|+\beta_j)$$
$$-(k_1-1)|\alpha_n|-(k_2-1)|\beta_n|+L+M-\alpha_\lambda-\beta_\mu].$$

Again

$$F(z)=a_0+G(z),$$

where

$$G(z)=-a_nz^{n+1}-(k_1-1)\alpha_nz^n+(k_1\alpha_n-\alpha_{n-1})z^n+(k_1\alpha_{n-1}-\alpha_{n-2})z^{n-1}+\ldots\ldots$$
$$+(k_1\alpha_{\lambda+1}-\alpha_\lambda)z^{\lambda+1}+(\alpha_\lambda-\alpha_{\lambda-1})z^\lambda+\ldots\ldots+(\alpha_1-\alpha_0)z$$
$$-(k_1-1)(\alpha_{n-1}z^{n-1}+\alpha_{n-2}z^{n-2}+\ldots\ldots.+\alpha_{\lambda+1}z^{\lambda+1})$$
$$+i\{(k_2\beta_n-\beta_{n-1})z^n-(k_2-1)\beta_nz^n+(k_2\beta_{n-1}-\beta_{n-2})z^{n-1}+\ldots\ldots$$
$$+(k_2\beta_{\mu+1}-\beta_\mu)z^{\mu+1}-(k_2-1)(\beta_{n-1}z^{n-1}+\ldots\ldots+\beta_{\mu+1}z^{\mu+1})+(\beta_\mu-\beta_{\mu-1})z^\mu$$
$$+\ldots\ldots+(\beta_1-\beta_0)z\}.$$

For $|z|=R, R>0$, we have

$$|G(z)|\le|a_n|R^{n+1}+(k_1-1)|\alpha_n|R^n+(k_1\alpha_n-\alpha_{n-1})R^n+(k_1\alpha_{n-1}-\alpha_{n-2})R^{n-1}+\ldots\ldots$$
$$+(k_1\alpha_{\lambda+1}-\alpha_\lambda)R^{\lambda+1}+|\alpha_\lambda-\alpha_{\lambda-1}|R^\lambda+\ldots\ldots+|\alpha_1-\alpha_0|R$$
$$+(k_1-1)(|\alpha_{n-1}|R^{n-1}+|\alpha_{n-2}|R^{n-2}+\ldots\ldots.+|\alpha_{\lambda+1}|R^{\lambda+1})$$
$$+(k_2\beta_n-\beta_{n-1})R^n+(k_2-1)|\beta_n|R^n+(k_2\beta_{n-1}-\beta_{n-2})R^{n-1}+\ldots\ldots$$
$$+(k_2\beta_{\mu+1}-\beta_\mu)R^{\mu+1}+(k_2-1)(|\beta_{n-1}|R^{n-1}+\ldots\ldots+|\beta_{\mu+1}|R^{\mu+1})+|\beta_\mu-\beta_{\mu-1}|R^\mu$$
$$+\ldots\ldots+|\beta_1-\beta_0|R\}$$
$$\le|a_n|R^{n+1}+R^n[(k_1-1)|\alpha_n|+k_1\alpha_n-\alpha_{n-1}+k_1\alpha_{n-1}-\alpha_{n-2}+\ldots\ldots+k_1\alpha_{\lambda+1}-\alpha_\lambda+L$$
$$+(k_1-1)(|\alpha_{n-1}|+\ldots\ldots+|\alpha_{\lambda+1}|)+k_2\beta_n-\beta_{n-1}$$
$$+k_2\beta_{n-1}-\beta_{n-2}+\ldots\ldots+k_2\beta_{\mu+1}-\beta_\mu+M+(k_2-1)(|\beta_{n-1}|+\ldots\ldots+|\beta_{\mu+1}|)]$$

$$=|a_n|R^{n+1}+R^n[(\alpha_n+\beta_n)-(\alpha_\lambda+\beta_\mu)+L+M-|\alpha_0|-|\beta_0|+(k_1-1)\sum_{j=\lambda+1}^{n}(|\alpha_j|+\alpha_j)$$
$$+(k_2-1)\sum_{j=\mu+1}^{n}(|\beta_j|+\beta_j)]$$
$$=X$$

for $R\ge1$ and
for $R\le1$

$$|G(z)|\le|a_n|R^{n+1}+R[(\alpha_n+\beta_n)-(\alpha_\lambda+\beta_\mu)+L+M-|\alpha_0|-|\beta_0|+(k_1-1)\sum_{j=\lambda+1}^{n}(|\alpha_j|+\alpha_j)$$
$$+(k_2-1)\sum_{j=\mu+1}^{n}(|\beta_j|+\beta_j)]$$
$$=Y$$

Since G(0)=0 and G(z) is analytic for $|z|\le R$, it follows, by Schwarz Lemma , that for $|z|\le R$,

$|G(z)| \le X|z|$ for $R \ge 1$ and $|G(z)| \le Y|z|$ for $R \le 1$.

Hence, for $|z| \le R$, $R \ge 1$

$$\begin{aligned} |F(z)| &= |a_0 + G(z)| \\ &\ge |a_0| - |G(z)| \\ &\ge |a_0| - X|z| \\ &> 0 \end{aligned}$$

if

$$|z| < \frac{|a_0|}{X}.$$

Similarly, for $|z| \le R$, $R \le 1$, $|F(z)| > 0$ if $|z| < \frac{|a_0|}{Y}$.

In other words, F(z) does not vanish in $|z| < \frac{|a_0|}{X}$ for $R \ge 1$ and F(z) does not vanish in $|z| < \frac{|a_0|}{Y}$ for $R \le 1$ in $|z| \le R$. That means all the zeros of F(z) and hence all the zeros of P(z) lie in $|z| \ge \frac{|a_0|}{X}$ for $R \ge 1$ and in $|z| \ge \frac{|a_0|}{Y}$ for $R \le 1$ in $|z| \le R$.

Since for $|z| \le R$, $|F(z)| \le X + |a_0|$ for $R \ge 1$ and $|F(z)| \le Y + |a_0|$ for $R \le 1$ and since $F(0) = a_0 \ne 0$, it follows by Lemma 2 that the number of zeros of P(z) in $\frac{|a_0|}{X} \le |z| \le \frac{R}{c}, c > 1, R \ge 1$ does not exceed $\frac{1}{\log c} \log \frac{X + |a_0|}{|a_0|} = \frac{1}{\log c} \log(1 + \frac{X}{|a_0|})$ and the number of zeros of P(z) in $\frac{|a_0|}{Y} \le |z| \le \frac{R}{c}, c > 1, R \le 1$ does not exceed $\frac{1}{\log c} \log \frac{Y + |a_0|}{|a_0|} = \frac{1}{\log c} \log(1 + \frac{Y}{|a_0|})$.

That proves Theorem 1 completely.

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