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Generating Functions of Certain Hypergeometric functions By Means of Fractional Calculus

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ABSTRACT
In the present investigation, we apply the fractional derivative techniques on some well-known identities to obtain linear generating functions for several classesof hypergeometric functions. Some special cases of results were also discussed in the end.
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I. INTRODUCTION

Fractional calculus is concerned with the theory of derivatives and integrals of non-integer order. There are many applications of fractional derivatives in the theory of hypergeometric functions, in solving ordinary and partial differential equations and integral equations (see [4], [5], [7], [12]). Nowadays this subject have wide application in several scientific areas like control theory, physics and engineering, stochastic process, modeling, probability theory etc.

In 1731, Euler extended the derivative formula ([12] pp.285), to the general form as

$$D_z^{\mu}\{z^{\lambda}\} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-n+1)} z^{\lambda-\mu} \tag{1.1}$$

whereμ is an ordinary complex number.

We recall here the application of Euler derivative formula to some special functions. For this we use the theorem.

Theorem 1: If a function $f(z)$ is analytic in the disc $|z|<\rho$, has the power series expansion,

$$f(z) = \sum_{n=0}^{\infty} (a)_n z^n, |z|<\rho \tag{1.2}$$

then ,

$$D_z^{\mu}\{z^{\lambda-1} f(z)\} = \sum_{n=0}^{\infty} (a)_n D_z^{\mu}\{z^{\lambda+\mu-1}\} = \frac{\Gamma(\lambda)}{\Gamma(\lambda-\mu)} z^{\lambda-\mu-1} \sum_{n=0}^{\infty} \frac{(a)_n (\lambda)_n}{(\lambda-\mu)_n} z^n \tag{1.3}$$

provided that $Re(\lambda)>0, Re(\mu)<0$, and $|z|<\rho$.

Some of the definition and notations used in the given manuscript are stated below:
Appellfunction of two variables defined by [1] are given as

$$F_1[a,b,b';\ c;\ x,y] = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m(b')_n x^m y^n}{(c)_{m+n} m!\, n!}, \tag{1.4}$$

$$F_2[a,b,b';\ c,c';\ x,y] = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m(b')_n x^m y^n}{(c)_m\ (c')_n m!\, n!}, \tag{1.5}$$

Generalization of Appell function of two variables by Khan M.A. and Abukhammash G.S. [2] are defined as

$$M_3(a,b,b',c,c';d,d',e,e';x,y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m(b')_n(c)_m(c')_n}{(d)_m(d')_n(e)_m(e')_n} \frac{x^m y^n}{m!\, n!} \tag{1.6}$$

$$M_4(a,b,b',c,c';d,e,e';x,y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m(b')_n(c)_m(c')_n}{(d)_{m+n}(e)_m(e')_n}\frac{x^m y^n}{m!\,n!} \tag{1.7}$$

$$M_7(a,b,c,c';d,e,e';x,y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}(c)_m(c')_n}{(d)_{m+n}(e)_m(e')_n}\frac{x^m y^n}{m!\,n!} \tag{1.8}$$

$$M_8(a,b,c,c';\ d,e;\ x,y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}(c)_m(c')_n}{(d)_{m+n}(e)_{m+n}}\frac{x^m y^n}{m!\,n!} \tag{1.9}$$

Lauricella [3], generalized the Appell double hypergeometric functions $F_1,\ldots,F_4$ to functions of n variables, but we use only two $F_A^{(n)}$ and $F_D^{(n)}$ are defined by

$$F_A^{(n)}[a,b_1,\ldots,b_n;c_1,\ldots,c_n;x_1,\ldots\ldots,x_n] = \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a)_{m_1+\ldots+m_n}(b_1)_{m_1}\cdots(b_n)_{m_n}}{(c_1)_{m_1}\cdots\cdots(c_n)_{m_n}}\frac{x_1^{m_1}}{m_1!}\cdots\cdots\frac{x_n^{m_n}}{m_n!} \tag{1.10}$$

$$|x_1|,\ldots\ldots\ldots\ldots,|x_n| < 1;$$

$$F_D^{(n)}[a,b_1,\ldots\ldots,b_n;c;x_1,\ldots\ldots,x_n] = \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a)_{m_1+\ldots+m_n}(b_1)_{m_1}\cdots\cdots(b_n)_{m_n}}{(c)_{m_1+\ldots+m_n}}\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!} \tag{1.11}$$

$$\max\{|x_1|,\ldots\ldots\ldots\ldots,|x_n|\} < 1.$$

In 1963, Pandey [6] established two interesting Horn's type hypergeometric functions of three variables, while transforming Pochhammer's double-loop contour integrals associated with the Lauricalla's functions F_G and F_F are given below:

$$G_A[\alpha,\beta,\beta';\gamma;x,y,z] = \sum_{m,n,p=0}^{\infty} \frac{(\alpha)_{n+p-m}(\beta)_{m+p}(\beta')_n}{(\gamma)_{n+p-m}\,m!\,n!\,p!}x^m y^n z^p \tag{1.12}$$

$$G_B[\alpha,\beta_1,\beta_2,\beta_3;\gamma;x,y,z] = \sum_{m,n,p=0}^{\infty} \frac{(\alpha)_{n+p-m}(\beta_1)_m(\beta_2)_n(\beta_3)_p}{(\gamma)_{n+p-m}\,m!\,n!\,p!}x^m y^n z^p \tag{1.13}$$

In this manuscript we use the following fractional derivative formulas [9] to obtain the several class of generating functions.

$$D_{x_1}^{\mu-\alpha}D_{x_2}^{\mu'-\alpha'}\left\{x_1{}^{-\alpha}x_2{}^{-\alpha'}\left(1-\frac{\omega_1}{x_1}-\frac{\omega_2}{x_2}\right)^{-\beta}\right\} = \frac{\Gamma(1-\alpha)\Gamma(1-\alpha')}{\Gamma(1-\mu)\Gamma(1-\mu')}x_1{}^{-\mu}x_2{}^{-\mu'}F_2\left[\beta,\mu,\mu';\alpha,\alpha';\frac{\omega_1}{x_1},\frac{\omega_2}{x_2}\right] \tag{1.14}$$

where, $\left|\frac{\omega_1}{x_1}+\frac{\omega_2}{x_2}\right| < 1$.

$$D_{x_1}^{\mu_1-\alpha_1}D_{x_2}^{\mu_2-\alpha_2}D_{x_3}^{\mu_3-\alpha_3}D_{x_4}^{\mu_4-\alpha_4}\left\{x_1{}^{-\alpha_1}x_2{}^{-\alpha_2}x_3{}^{-\alpha_3}x_4{}^{-\alpha_4}\left(1-\frac{\omega_1}{x_1x_2}-\frac{\omega_2}{x_3x_4}\right)^{-\beta}\right\}$$
$$= \frac{\Gamma(1-\alpha_1)\Gamma(1-\alpha_2)\Gamma(1-\alpha_3)\Gamma(1-\alpha_4)}{\Gamma(1-\mu_1)\Gamma(1-\mu_2)\Gamma(1-\mu_3)\Gamma(1-\mu_4)}x_1{}^{-\mu_1}x_2{}^{-\mu_2}x_3{}^{-\mu_3}x_4{}^{-\mu_4}$$
$$\times M_3\left[\beta,\mu_1,\mu_2,\mu_3,\mu_4;\alpha_1,\alpha_2,\alpha_3,\alpha_4;\frac{\omega_1}{x_1x_2},\frac{\omega_2}{x_3x_4}\right] \tag{1.15}$$

where, $\left|\frac{\omega_1}{x_1x_2}+\frac{\omega_2}{x_3x_4}\right| < 1$.

$$D_{x_1}^{\mu_1-\alpha_1}D_{x_2}^{\mu_2-\alpha_2}D_{x_3}^{\mu_3-\alpha_3}\left\{x_1{}^{-\alpha_1}x_2{}^{-\alpha_2}x_3{}^{-\alpha_3}\left(1-\frac{\omega_1}{x_1x_2}-\frac{\omega_2}{x_1x_3}\right)^{-\beta}\right\}$$
$$= \frac{\Gamma(1-\alpha_1)\Gamma(1-\alpha_2)\Gamma(1-\alpha_3)}{\Gamma(1-\mu_1)\Gamma(1-\mu_2)\Gamma(1-\mu_3)}x_1{}^{-\mu_1}x_2{}^{-\mu_2}x_3{}^{-\mu_3}M_7\left[\beta,\mu_1,\mu_2,\mu_3;\alpha_1,\alpha_2,\alpha_3;\frac{\omega_1}{x_1x_2},\frac{\omega_2}{x_1x_3}\right] \tag{1.16}$$

where, $\left|\frac{\omega_1}{x_1x_2}+\frac{\omega_2}{x_1x_3}\right| < 1$.

$$D_x^{\alpha-\mu}\left\{x^\alpha(1-x)^{-\beta}\left(1-\omega_1x-\frac{\omega_2}{1-x}\right)^{-\gamma}\right\} = \frac{\Gamma(1+\alpha)}{\Gamma(1+\mu)}x^\mu H_A[\beta,\gamma,1+\alpha;\beta,1+\mu;\omega_2,\omega_1x,x] \tag{1.17}$$

where, $Re(\alpha) \geq 0, |x| < 1, \left|\omega_1x+\frac{\omega_2}{1-x}\right| < 1$.

$$D_x^{\alpha-\mu}\left\{x^\alpha(1-x)^{-\beta}\left(1-\frac{\omega x}{1-x}\right)^{-\gamma}\right\}$$
$$= \frac{\Gamma(1+\alpha)}{\Gamma(1+\mu)}x^\mu(1-x)^{-\alpha-1}F_1\left[1+\alpha,\gamma,1+\mu-\beta;1+\mu;\frac{\omega x}{1-x},\frac{-x}{1-x}\right] \tag{1.18}$$

where, $Re(\alpha) \geq 0, |x| < 1, \left|\frac{\omega x}{1-x}\right| < 1$.

$$D_x^{\alpha-\mu}\left\{x^\alpha (1-x)^{-\beta}\left(1-\frac{\omega_1 x}{1-x}\right)^{-\gamma}\left(1-\frac{\omega_2 x}{1-x}\right)^{-\delta}\right\}$$
$$=\frac{\Gamma(1+\alpha)}{\Gamma(1+\mu)}x^\mu (1-x)^{-\alpha-1}F_D^{(3)}\left[1+\alpha,\gamma,\delta,1+\mu-\beta;1+\mu;\frac{\omega_1 x}{1-x},\frac{\omega_2 x}{1-x},\frac{x}{x-1}\right] \quad (1.19)$$

where, $Re(\alpha) \geq 0, |x| < 1, \left|\frac{\omega_1 x}{1-x}\right| < 1, \left|\frac{\omega_2 x}{1-x}\right| < 1$.

$$D_x^{\alpha-\mu}\left\{x^\alpha (1-x)^{-\beta}(1-\omega_1 x)^{-\gamma}\left(1-\frac{\omega_2}{1-x}\right)^{-\delta}\right\}$$
$$=\frac{\Gamma(1+\alpha)}{\Gamma(1+\mu)}x^\mu F_M[\delta,1+\alpha,1+\alpha,\beta,\gamma,\beta;\beta,1+\mu,1+\mu;\omega_2,\omega_1 x,x] \quad (1.20)$$

where, $Re(\alpha) \geq 0, |x| < 1, |\omega_1 x| < 1, \left|\frac{\omega_2}{1-x}\right| < 1$.

$$D_x^{\alpha-\mu}\left\{x^\alpha (1-x)^{-\beta}(1-\omega_1 x)^{-\gamma}\left(1-\frac{\omega_2 x}{1-x}\right)^{-\delta}\right\}$$
$$=\frac{\Gamma(1+\alpha)}{\Gamma(1+\mu)}x^\mu F^{(3)}\left[\begin{matrix}1+\alpha::--;\ \beta;--:\gamma;\ \delta;--;\\1+\mu::--;--;--:--;\beta;--;\end{matrix}\omega_1 x,\omega_2 x,x\right] \quad (1.21)$$

where, $Re(\alpha) \geq 0, |x| < 1, |\omega_1 x| < 1, \left|\frac{\omega_2 x}{1-x}\right| < 1$.

$$D_{x_1}^{\mu_1-\alpha_1}D_{x_2}^{\mu_2-\alpha_2}D_{x_3}^{\mu_3-\alpha_3}\left\{x_1{}^{-\alpha_1}x_2{}^{-\alpha_2}x_3{}^{-\alpha_3}\left(1-\frac{\omega_1}{x_1x_2}\right)^{-\beta}\left(1-\frac{\omega_2}{x_1x_3}\right)^{-\gamma}\right\}$$
$$=\frac{\Gamma(1-\alpha_1)\Gamma(1-\alpha_2)\Gamma(1-\alpha_3)}{\Gamma(1-\mu_1)\Gamma(1-\mu_2)\Gamma(1-\mu_3)}x_1{}^{-\mu_1}x_2{}^{-\mu_2}x_3{}^{-\mu_3}M_4\left[\mu_1,\beta,\gamma,\mu_2,\mu_3;\alpha_1,\alpha_2,\alpha_3;\frac{\omega_1}{x_1x_2},\frac{\omega_2}{x_1x_3}\right] \quad (1.22)$$

where, $\left|\frac{\omega_1}{x_1x_2}\right| < 1, \left|\frac{\omega_2}{x_1x_3}\right| < 1$.

$$D_{x_1}^{\alpha-\mu}D_{x_2}^{\alpha'-\mu'}\{x_1{}^\alpha x_2{}^{\alpha'}(1-\omega_1x_1x_2)^{-\beta}(1-\omega_2x_1x_2)^{-\gamma}\}$$
$$=\frac{\Gamma(1+\alpha)\Gamma(1+\alpha')}{\Gamma(1+\mu)\Gamma(1+\mu')}x_1{}^\mu x_2{}^\mu M_8[1+\alpha,1+\alpha',\beta,\gamma;1+\mu,1+\mu';\omega_1x_1x_2,\omega_2x_1x_2] \quad (1.23)$$

where, $Re(\alpha) \geq 0, Re(\alpha') \geq 0, |\omega_1x_1x_2| < 1, |\omega_2x_1x_2| < 1$.

$$D_x^{\alpha-\mu}\left\{x^\alpha(1-\omega_1x)^{-\beta}\left(1-\omega_2x-\frac{\omega_3}{x}\right)^{-\gamma}\right\}=\frac{\Gamma(1+\alpha)}{\Gamma(1+\mu)}x^\mu G_A\left[1+\alpha,\gamma,\beta;1+\mu;\omega_1x,\omega_2x,\frac{\omega_3}{x}\right] \quad (1.24)$$

where, $Re(\alpha) \geq 0, |\omega_1 x| < 1, \left|\omega_2 x+\frac{\omega_3}{x}\right| < 1$.

$$D_x^{\mu-\alpha}\left\{x^{-\alpha}\left(1-\frac{\omega_1}{x}\right)^{-\beta}\left(1-\frac{\omega_2}{x}\right)^{-\gamma}\right\}=\frac{\Gamma(1-\alpha)}{\Gamma(1-\mu)}x^{-\mu}F_1\left[\mu,\beta,\gamma;\alpha;\frac{\omega_1}{x},\frac{\omega_2}{x}\right] \quad (1.25)$$

where, $\left|\frac{\omega_1}{x}\right| < 1, \left|\frac{\omega_2}{x}\right| < 1$.

$$D_x^{\alpha-\mu}\left\{x^\alpha(1-\omega_1x)^{-\beta}(1-\omega_2x)^{-\gamma}\left(1-\frac{\omega_3}{x}\right)^{-\delta}\right\}$$
$$=\frac{\Gamma(1+\alpha)}{\Gamma(1+\mu)}x^\mu G_B\left[1+\alpha,\delta,\beta,\gamma;1+\mu;\frac{\omega_3}{x},\omega_1x,\omega_2x\right] \quad (1.26)$$

where, $Re(\alpha) \geq 0, |\omega_1 x| < 1, |\omega_2 x| < 1, \left|\frac{\omega_3}{x}\right| < 1$.

$$D_x^{\alpha-\mu}\{x^\alpha(1-\omega_1x)^{-\beta}(1-\omega_2x)^{-\gamma}(1-\omega_3x)^{-\delta}\}$$
$$=\frac{\Gamma(1+\alpha)}{\Gamma(1+\mu)}x^\mu F_D^{(3)}[1+\alpha,\beta,\gamma,\delta;1+\mu;\omega_1x,\omega_2x,\omega_3x] \quad (1.27)$$

where, $Re(\alpha) \geq 0, |\omega_1 x| < 1, |\omega_2 x| < 1, |\omega_3 x| < 1$.

II. LINEAR GENERATING FUNCTIONS

Consider the elementary identity (see, [12], Sec. 5.2, Eq. 1),

$$[(1-x)-t]^{-\lambda}=(1-t)^{-\lambda}\left\{1-\frac{x}{1-t}\right\}^{-\lambda} \quad (2.1)$$

can be written as

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}(1-x)^{-(\lambda+n)}t^n=(1-t)^{-\lambda}\left[1-\frac{x}{1-t}\right]^{-\lambda}, |t| < |1-x|. \quad (2.2)$$

Replace x by $\frac{\omega_1}{x_1}+\frac{\omega_2}{x_2}$, multiply both side of (2.2) by ${x_1}^{-\alpha}{x_2}^{-\alpha'}$ and then apply the fractional derivative operator $D_{x_1}^{\mu-\alpha}D_{x_2}^{\mu'-\alpha'}$, to obtain

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}D_{x_1}^{\mu-\alpha}D_{x_2}^{\mu'-\alpha'}\left\{{x_1}^{-\alpha}{x_2}^{-\alpha'}\left(1-\frac{\omega_1}{x_1}-\frac{\omega_2}{x_2}\right)^{-(\lambda+n)}\right\}t^n$$

$$=(1-t)^{-\lambda}D_{x_1}^{\mu-\alpha}D_{x_2}^{\mu'-\alpha'}\left\{{x_1}^{-\alpha}{x_2}^{-\alpha'}\left(1-\frac{\omega_1}{x_1(1-t)}-\frac{\omega_2}{x_2(1-t)}\right)^{-\lambda}\right\} \tag{2.3}$$

with the aid of the relation (1.14), equation (2.3) yield's the generating relation,

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}F_2\left[\lambda+n,\mu,\mu';\alpha,\alpha';\frac{\omega_1}{x_1},\frac{\omega_2}{x_2}\right]t^n$$

$$=(1-t)^{-\lambda}F_2\left[\lambda,\mu,\mu';\alpha,\alpha';\frac{\omega_1}{x_1(1-t)},\frac{\omega_2}{x_2(1-t)}\right] \tag{2.4}$$

Similarly, the generalization of the generating function (2.4) can be obtained in the following form:

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}F_A^{(n)}\left[\lambda+n,\mu_1,\ldots\ldots,\mu_n;\alpha_1,\ldots..\alpha_n;\frac{\omega_1}{x_1},\ldots\ldots,\frac{\omega_n}{x_n}\right]t^n$$

$$=(1-t)^{-\lambda}F_A^{(n)}\left[\lambda,\mu_1,\ldots\ldots,\mu_n;\alpha_1,\ldots.\alpha_n;\frac{\omega_1}{x_1(1-t)},\ldots\ldots,\frac{\omega_n}{x_n(1-t)}\right] \tag{2.5}$$

Replace x by $\frac{\omega_1}{x_1x_2}+\frac{\omega_2}{x_3x_4}$, multiply both side of (2.2) by ${x_1}^{-\alpha_1}{x_2}^{-\alpha_2}{x_3}^{-\alpha_3}{x_4}^{-\alpha_4}$ and then apply the fractional derivative operator $D_{x_1}^{\mu_1-\alpha_1}D_{x_2}^{\mu_2-\alpha_2}D_{x_3}^{\mu_3-\alpha_3}D_{x_4}^{\mu_4-\alpha_4}$, to obtain

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}D_{x_1}^{\mu_1-\alpha_1}D_{x_2}^{\mu_2-\alpha_2}D_{x_3}^{\mu_3-\alpha_3}D_{x_4}^{\mu_4-\alpha_4}\left\{{x_1}^{-\alpha_1}{x_2}^{-\alpha_2}{x_3}^{-\alpha_3}{x_4}^{-\alpha_4}\left(1-\frac{\omega_1}{x_1x_2}-\frac{\omega_2}{x_3x_4}\right)^{-(\lambda+n)}\right\}t^n$$

$$=(1-t)^{-\lambda}D_{x_1}^{\mu_1-\alpha_1}D_{x_2}^{\mu_2-\alpha_2}D_{x_3}^{\mu_3-\alpha_3}D_{x_4}^{\mu_4-\alpha_4}\left\{{x_1}^{-\alpha_1}{x_2}^{-\alpha_2}{x_3}^{-\alpha_3}{x_4}^{-\alpha_4}\left(1-\frac{\omega_1}{x_1x_2(1-t)}-\frac{\omega_2}{x_3x_4(1-t)}\right)^{-\lambda}\right\} \tag{2.6}$$

with the aid of the relation (1.15), equation (2.6) yield's the generating relation,

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}M_3\left[\lambda+n,\mu_1,\mu_2,\mu_3,\mu_4;\alpha_1,\alpha_2,\alpha_3,\alpha_4;\frac{\omega_1}{x_1x_2},\frac{\omega_2}{x_3x_4}\right]t^n$$

$$=(1-t)^{-\lambda}M_3\left[\lambda,\mu_1,\mu_2,\mu_3,\mu_4;\alpha_1,\alpha_2,\alpha_3,\alpha_4;\frac{\omega_1}{x_1x_2(1-t)},\frac{\omega_2}{x_3x_4(1-t)}\right] \tag{2.7}$$

Replace x by $\frac{\omega_1}{x_1x_2}+\frac{\omega_2}{x_1x_3}$, multiply both side of (2.2) by ${x_1}^{-\alpha_1}{x_2}^{-\alpha_2}{x_3}^{-\alpha_3}$ and then apply the fractional derivative operator $D_{x_1}^{\mu_1-\alpha_1}D_{x_2}^{\mu_2-\alpha_2}D_{x_3}^{\mu_3-\alpha_3}$, to obtain

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}D_{x_1}^{\mu_1-\alpha_1}D_{x_2}^{\mu_2-\alpha_2}D_{x_3}^{\mu_3-\alpha_3}\left\{{x_1}^{-\alpha_1}{x_2}^{-\alpha_2}{x_3}^{-\alpha_3}\left(1-\frac{\omega_1}{x_1x_2}-\frac{\omega_2}{x_1x_3}\right)^{-(\lambda+n)}\right\}t^n$$

$$=(1-t)^{-\lambda}D_{x_1}^{\mu_1-\alpha_1}D_{x_2}^{\mu_2-\alpha_2}D_{x_3}^{\mu_3-\alpha_3}\left\{{x_1}^{-\alpha_1}{x_2}^{-\alpha_2}{x_3}^{-\alpha_3}\left(1-\frac{\omega_1}{x_1x_2(1-t)}-\frac{\omega_2}{x_1x_3(1-t)}\right)^{-\lambda}\right\} \tag{2.8}$$

with the aid of the relation (1.16), equation (2.8) yield's the generating relation,

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}M_7\left[\lambda+n,\mu_1,\mu_2,\mu_3;\alpha_1,\alpha_2,\alpha_3;\frac{\omega_1}{x_1x_2},\frac{\omega_2}{x_1x_3}\right]t^n$$

$$=(1-t)^{-\lambda}M_7\left[\lambda,\mu_1,\mu_2,\mu_3;\alpha_1,\alpha_2,\alpha_3;\frac{\omega_1}{x_1x_2(1-t)},\frac{\omega_2}{x_1x_3(1-t)}\right] \tag{2.9}$$

Replace x by $\omega_1x+\frac{\omega_2}{1-x}$, multiply both side of (2.2) by $x^{\alpha}(1-x)^{-\beta}$ and then on applying the fractional derivative operator $D_x^{\alpha-\mu}$, to obtain

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}D_x^{\alpha-\mu}\left\{x^{\alpha}(1-x)^{-\beta}\left(1-\omega_1x-\frac{\omega_2}{1-x}\right)^{-(\lambda+n)}\right\}t^n$$

$$=(1-t)^{-\lambda}D_x^{\alpha-\mu}\left\{x^{\alpha}(1-x)^{-\beta}\left(1-\frac{\omega_1x}{(1-t)}-\frac{\omega_2}{(1-x)(1-t)}\right)^{-\lambda}\right\} \tag{2.10}$$

with the aid of the relation (1.17), equation (2.10) yield's the generating relation,

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}H_A[\beta,\lambda+n,1+\alpha;\beta,1+\mu;\omega_2,\omega_1x,x]t^n$$

$$= (1-t)^{-\lambda} H_A\left[\beta, \lambda, 1+\alpha; \beta, 1+\mu; \frac{\omega_2}{(1-t)}, \frac{\omega_1 x}{(1-t)}, x\right] \quad (2.11)$$

where hypergeometric function H_A is defined by Srivastava [10] as

$$H_A[\alpha, \beta, \beta'; \gamma, \gamma'; x, y, z] = \sum_{m,n,p=0}^{\infty} \frac{(\alpha)_{m+n+p}(\beta)_{m+n}(\beta')_{n+p}}{(\gamma)_m(\gamma')_{n+p} m!\, n!\, p!} x^m y^n z^p \quad (2.12)$$

$$|x| < r, |y| < s, |z| < t, r+s+t = 1+st.$$

The analysis used to obtain the relation (2.11) is further employed to obtain linear generating function by using the relation (1.18), is given below:

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} F_1\left[1+\alpha, \lambda+n, 1+\mu-\beta; 1+\mu; \frac{\omega x}{1-x}, \frac{x}{x-1}\right] t^n$$

$$= (1-t)^{-\lambda} F_1\left[1+\alpha, \lambda, 1+\mu-\beta; 1+\mu; \frac{\omega x}{(1-x)(1-t)}, \frac{x}{x-1}\right] \quad (2.13)$$

Further, in (2.2), replace x by $\frac{\omega_1 x}{1-x}, \frac{\omega_2 x}{1-x}$, t by t_1, t_2 and λ by λ_1, λ_2 respectively. Then multiply the two equation, to obtain

$$\sum_{m,n=0}^{\infty} \frac{(\lambda_1)_m(\lambda_2)_n}{m!\, n!}\left(1-\frac{\omega_1 x}{1-x}\right)^{-(\lambda_1+m)}\left(1-\frac{\omega_2 x}{1-x}\right)^{-(\lambda_2+n)} (t_1)^m (t_2)^n$$

$$= (1-t_1)^{-\lambda_1}(1-t_2)^{-\lambda_2}\left(1-\frac{\omega_1 x}{(1-x)(1-t_1)}\right)^{-\lambda_1}\left(1-\frac{\omega_2 x}{(1-x)(1-t_2)}\right)^{-\lambda_2} \quad (2.14)$$

Now multiply both sides of (2.14) by $x^{\alpha}(1-x)^{-\beta}$ and then by appealing the fractional derivative operator $D_x^{\alpha-\mu}$ on both sides and using (1.19), one obtains the double sum generating relation as

$$\sum_{m,n=0}^{\infty} \frac{(\lambda_1)_m(\lambda_2)_n}{m!\, n!}(t_1)^m (t_2)^n F_D^{(3)}\left[1+\alpha, \lambda_1+m, \lambda_2+n, 1+\mu-\beta; 1+\mu; \frac{\omega_1 x}{1-x}, \frac{\omega_2 x}{1-x}, \frac{x}{x-1}\right]$$

$$= (1-t_1)^{-\lambda_1}(1-t_2)^{-\lambda_2} F_D^{(3)}\left[1+\alpha, \lambda_1, \lambda_2, 1+\mu-\beta; 1+\mu; \frac{\omega_1 x}{(1-x)(1-t_1)}, \frac{\omega_2 x}{(1-x)(1-t_2)}, \frac{x}{x-1}\right] \quad (2.15)$$

where $F_D^{(3)}$ is defined by (1.11) at $n = 3$.

Further, we adopt the analysis similar to (2.15) and use the relations (1.20), (1.21), respectively, yield's the following double sum generating functions as follows:

$$\sum_{m,n=0}^{\infty} \frac{(\lambda_1)_m(\lambda_2)_n}{m!\, n!}(t_1)^m (t_2)^n F_M[\lambda_2+n, 1+\alpha, 1+\alpha, \beta, \lambda_1+m, \beta; \beta, 1+\mu, 1+\mu; \omega_2, \omega_1 x, x]$$

$$= (1-t_1)^{-\lambda_1}(1-t_2)^{-\lambda_2} F_M\left[\lambda_2, 1+\alpha, 1+\alpha, \beta, \lambda_1, \beta; \beta, 1+\mu, 1+\mu; \frac{\omega_2}{(1-t_2)}, \frac{\omega_1 x}{(1-t_1)}, x\right] \quad (2.16)$$

where F_M is defined by [8] as

$$F_{11}: F_M[\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_2, \gamma_2; x, y, z] = \sum_{m,n,p=0}^{\infty} \frac{(\alpha_1)_m(\alpha_2)_{n+p}(\beta_1)_{m+p}(\beta_2)_n}{(\gamma_1)_m(\gamma_2)_{n+p} m!\, n!\, p!} x^m y^n z^p \quad (2.17)$$

$$\sum_{m,n=0}^{\infty} \frac{(\lambda_1)_m(\lambda_2)_n}{m!\, n!}(t_1)^m (t_2)^n F^{(3)}\left[\begin{matrix} 1+\alpha::-;\beta;-:\lambda_1+m;\lambda_2+n;-; \\ 1+\mu::--;--;--:--;\beta;--; \end{matrix} \omega_1 x, \omega_2 x, x\right]$$

$$= (1-t_1)^{-\lambda_1}(1-t_2)^{-\lambda_2} F^{(3)}\left[\begin{matrix} 1+\alpha::-;\beta;-:\lambda_1;\lambda_2;-; \\ 1+\mu::-;-;-:-;\beta;-; \end{matrix} \frac{\omega_1 x}{(1-t_1)}, \frac{\omega_2 x}{(1-t_2)}, x\right] \quad (2.18)$$

where the general triple hypergeometric series $F^{(3)}[x, y, z]$is defined by [11] as

$$F^{(3)}[x, y, z] = F^{(3)}\left[\begin{matrix} (a)::(b);(b');(b''):(c);(c');(c''); \\ (e)::(g);(g');(g''):(h);(h');(h''); \end{matrix} x, y, z\right]$$

$$= \sum_{m,n,p=0}^{\infty} \Lambda(m,n,p) \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!} \quad (2.19)$$

where, for convenience,

$$\Lambda(m,n,p) = \frac{\prod_{j=1}^{A}(a_j)_{m+n+p}\prod_{j=1}^{B}(b_j)_{m+n}\prod_{j=1}^{B'}(b'_j)_{n+p}\prod_{j=1}^{B''}(b''_j)_{p+m}}{\prod_{j=1}^{E}(e_j)_{m+n+p}\prod_{j=1}^{G}(g_j)_{m+n}\prod_{j=1}^{G'}(g'_j)_{n+p}\prod_{j=1}^{G''}(g''_j)_{p+m}}$$

$$\times \frac{\prod_{j=1}^{C}(c_j)_m\prod_{j=1}^{C'}(c'_j)_n\prod_{j=1}^{C''}(c''_j)_p}{\prod_{j=1}^{H}(h_j)_m\prod_{j=1}^{H'}(h'_j)_n\prod_{j=1}^{H''}(h''_j)_p}$$

Again, in (2.2), replace x by $\frac{\omega_1}{x_1x_2}, \frac{\omega_2}{x_1x_3}$, t by t_1, t_2 and λ by λ_1, λ_2 respectively. Then multiply the two equation, to obtain

$$\sum_{m,n=0}^{\infty} \frac{(\lambda_1)_m(\lambda_2)_n}{m!\,n!}\left(1-\frac{\omega_1}{x_1x_2}\right)^{-(\lambda_1+m)}\left(\frac{1-\omega_2}{x_1x_3}\right)^{-(\lambda_2+n)}(t_1)^m(t_2)^n$$

$$= (1-t_1)^{-\lambda_1}(1-t_2)^{-\lambda_2}\left(1-\frac{\omega_1}{x_1x_2(1-t_1)}\right)^{-\lambda_1}\left(1-\frac{\omega_2}{x_1x_3(1-t_2)}\right)^{-\lambda_2} \quad (2.20)$$

Now multiply both sides of (2.20) by ${x_1}^{-\alpha_1}{x_2}^{-\alpha_2}{x_3}^{-\alpha_3}$ and then by appealing the fractional derivative operator $D_{x_1}^{\mu_1-\alpha_1}D_{x_2}^{\mu_2-\alpha_2}D_{x_3}^{\mu_3-\alpha_3}$ on both sides and using (1.22), one obtains the double sum generating relation as

$$\sum_{m,n=0}^{\infty} \frac{(\lambda_1)_m(\lambda_2)_n}{m!\,n!} M_4\left[\mu_1,\lambda_1+m,\lambda_2+n,\mu_2,\mu_3;\alpha_1,\alpha_2,\alpha_3;\frac{\omega_1}{x_1x_2},\frac{\omega_2}{x_1x_3}\right](t_1)^m(t_2)^n$$

$$= (1-t_1)^{-\lambda_1}(1-t_2)^{-\lambda_2}M_4\left[\mu_1,\lambda_1,\lambda_2,\mu_2,\mu_3;\alpha_1,\alpha_2,\alpha_3;\frac{\omega_1}{x_1x_2(1-t_1)},\frac{\omega_2}{x_1x_3(1-t_2)}\right] \quad (2.21)$$

Also, in (2.2), replace x by $1-\omega_1x_1x_2$, $1-\omega_2x_1x_2$, t by t_1, t_2 and λ by λ_1, λ_2 respectively. Then multiply the two equation, to obtain

$$\sum_{m,n=0}^{\infty} \frac{(\lambda_1)_m(\lambda_2)_n}{m!\,n!}(1-\omega_1x_1x_2)^{-(\lambda_1+m)}(1-\omega_2x_1x_2)^{-(\lambda_2+n)}(t_1)^m(t_2)^n$$

$$= (1-t_1)^{-\lambda_1}(1-t_2)^{-\lambda_2}\left(1-\frac{\omega_1x_1x_2}{(1-t_1)}\right)^{-\lambda_1}\left(1-\frac{\omega_2x_1x_2}{(1-t_2)}\right)^{-\lambda_2} \quad (2.22)$$

Now multiply both sides of (2.22) by ${x_1}^{\alpha_1}{x_2}^{\alpha_2}$ and then by appealing the fractional derivative operator $D_{x_1}^{\alpha_1-\mu_1}D_{x_2}^{\alpha_2-\mu_2}$ on both sides and using (1.23), one obtains the double sum generating relation as

$$\sum_{m,n=0}^{\infty} \frac{(\lambda_1)_m(\lambda_2)_n}{m!\,n!}(t_1)^m(t_2)^n M_8[1+\alpha_1,1+\alpha_2,\lambda_1+m,\lambda_2+n;1+\mu_1,1+\mu_2;\omega_1x_1x_2,\omega_2x_1x_2]$$

$$= (1-t_1)^{-\lambda_1}(1-t_2)^{-\lambda_2}M_8\left[1+\alpha_1,1+\alpha_2,\lambda_1,\lambda_2;1+\mu_1,1+\mu_2;\frac{\omega_1x_1x_2}{(1-t_1)},\frac{\omega_2x_1x_2}{(1-t_2)}\right] \quad (2.23)$$

Also, in (2.2), replace x by $x\omega_1$, $x\omega_2+\frac{\omega_3}{x}$, t by t_1, t_2 and λ by λ_1, λ_2 respectively. Then multiply the two equation, to obtain

$$\sum_{m,n=0}^{\infty} \frac{(\lambda_1)_m(\lambda_2)_n}{m!\,n!}(1-\omega_1x)^{-(\lambda_1+m)}\left(1-\omega_2x-\frac{\omega_3}{x}\right)^{-(\lambda_2+n)}(t_1)^m(t_2)^n$$

$$= (1-t_1)^{-\lambda_1}(1-t_2)^{-\lambda_2}\left(1-\frac{\omega_1x}{(1-t_1)}\right)^{-\lambda_1}\left(1-\frac{\omega_2x}{(1-t_2)}-\frac{\omega_3}{x(1-t_2)}\right)^{-\lambda_2} \quad (2.24)$$

Now multiply both sides of (2.24) by x^{α} and then by appealing the fractional derivative operator $D_x^{\alpha-\mu}$ on both sides and using (1.24), one obtains the double sum generating relation as

$$\sum_{m,n=0}^{\infty} \frac{(\lambda_1)_m(\lambda_2)_n}{m!\,n!} G_A\left[1+\alpha,\lambda_2+n,\lambda_1+m;1+\mu;\omega_1x,\omega_2x,\frac{\omega_3}{x}\right](t_1)^m(t_2)^n$$

$$= (1-t_1)^{-\lambda_1}(1-t_2)^{-\lambda_2}G_A\left[1+\alpha,\lambda_2,\lambda_1;1+\mu;\frac{\omega_1x}{(1-t_1)},\frac{\omega_2x}{(1-t_2)},\frac{\omega_3}{x(1-t_2)}\right] \quad (2.25)$$

Also, in (2.2), replace x by $\frac{\omega_1}{x}, \frac{\omega_2}{x}$, t by t_1, t_2 and λ by λ_1, λ_2 respectively. Then multiply the two equation, to obtain

$$\sum_{m,n=0}^{\infty} \frac{(\lambda_1)_m(\lambda_2)_n}{m!\,n!}\left(1-\frac{\omega_1}{x}\right)^{-(\lambda_1+m)}\left(1-\frac{\omega_2}{x}\right)^{-(\lambda_2+n)}(t_1)^m(t_2)^n$$

$$= (1-t_1)^{-\lambda_1}(1-t_2)^{-\lambda_2}\left(1-\frac{\omega_1}{x(1-t_1)}\right)^{-\lambda_1}\left(1-\frac{\omega_2}{x(1-t_2)}\right)^{-\lambda_2} \quad (2.26)$$

Now multiply both sides of (2.26) by $x^{-\alpha}$ and then by appealing the fractional derivative operator $D_x^{\mu-\alpha}$ on both sides and using (1.25), one obtains the double sum generating relation as

$$\sum_{m,n=0}^{\infty} \frac{(\lambda_1)_m(\lambda_2)_n}{m!\,n!} F_1\left[\mu,\lambda_1+m,\lambda_2+n;\alpha;\frac{\omega_1}{x},\frac{\omega_2}{x}\right](t_1)^m(t_2)^n$$

$$= (1-t_1)^{-\lambda_1}(1-t_2)^{-\lambda_2} F_1\left[\mu,\lambda_1,\lambda_2;\alpha;\frac{\omega_1}{x(1-t_1)},\frac{\omega_2}{x(1-t_2)}\right] \quad (2.27)$$

Similarly, the generalization of the generating function (2.27) can be obtained in the following form:

$$\sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(\lambda_1)_{m_1}\ldots\ldots(\lambda_n)_{m_n}}{(m_1)!\ldots\ldots(m_n)!} F_D^{(n)}\left[\mu,\lambda_1+m_1,\ldots\ldots,\lambda_n+m_n;\alpha;\frac{\omega_1}{x},\ldots\ldots,\frac{\omega_n}{x}\right](t_1)^{m_1}\ldots\ldots(t_n)^{m_n}$$

$$= (1-t_1)^{-\lambda_1}\ldots(1-t_n)^{-\lambda_n} F_D^{(n)}\left[\mu,\lambda_1,\ldots.,\lambda_n;\alpha;\frac{\omega_1}{x(1-t_1)},\ldots.,\frac{\omega_n}{x(1-t_n)}\right] \quad (2.28)$$

Replace x by $\omega_1 x$, $\omega_2 x$, $\frac{\omega_3}{x}$, t by t_1, t_2, t_3 and λ by λ_1, λ_2, λ_3 in (2.2) respectively. Then multiply the three equation, to obtain

$$\sum_{m,n,p=0}^{\infty} \frac{(\lambda_1)_m(\lambda_2)_n(\lambda_2)_p}{m!\,n!\,p!}(1-\omega_1 x)^{-(\lambda_1+m)}(1-\omega_2 x)^{-(\lambda_2+n)}\left(1-\frac{\omega_3}{x}\right)^{-(\lambda_3+p)}(t_1)^m(t_2)^n(t_3)^p$$

$$= (1-t_1)^{-\lambda_1}(1-t_2)^{-\lambda_2}(1-t_3)^{-\lambda_3}\left(1-\frac{\omega_1 x}{(1-t_1)}\right)^{-\lambda_1}\left(1-\frac{\omega_2 x}{(1-t_2)}\right)^{-\lambda_2}\left(1-\frac{\omega_3}{x(1-t_3)}\right)^{-\lambda_3} \quad (2.29)$$

Now multiply both sides of (2.29) by x^α and then by appealing the fractional derivative operator $D_x^{\alpha-\mu}$ on both sides and using (1.26), one obtains the triple sum generating relation as

$$\sum_{m,n,p=0}^{\infty} \frac{(\lambda_1)_m(\lambda_2)_n(\lambda_2)_p}{m!\,n!\,p!} G_B\left[1+\alpha,\lambda_3+p,\lambda_1+m,\lambda_2+n;1+\mu;\frac{\omega_3}{x},\omega_1 x,\omega_2 x\right](t_1)^m(t_2)^n(t_3)^p$$

$$= (1-t_1)^{-\lambda_1}(1-t_2)^{-\lambda_2}(1-t_3)^{-\lambda_3} G_B\left[1+\alpha,\lambda_3,\lambda_1,\lambda_2;1+\mu;\frac{\omega_3}{x(1-t_3)},\frac{\omega_1 x}{(1-t_1)},\frac{\omega_2 x}{(1-t_2)}\right] \quad (2.30)$$

Further, we adopt the analysis similar to (2.30) and use the relations (1.27), we obtain the following triple sum generating function as follows:

$$\sum_{m,n,p=0}^{\infty} \frac{(\lambda_1)_m(\lambda_2)_n(\lambda_2)_p}{m!\,n!\,p!}(t_1)^m(t_2)^n(t_3)^p F_D^{(3)}[1+\alpha,\lambda_1+m,\lambda_2+n,\lambda_3+p;1+\mu;\omega_1 x,\omega_2 x,\omega_3 x]$$

$$= (1-t_1)^{-\lambda_1}(1-t_2)^{-\lambda_2}(1-t_3)^{-\lambda_3} F_D^{(3)}\left[1+\alpha,\lambda_1,\lambda_2,\lambda_3;1+\mu;\frac{\omega_1 x}{(1-t_1)},\frac{\omega_2 x}{(1-t_2)},\frac{\omega_3 x}{(1-t_3)}\right] \quad (2.31)$$

III. SPECIAL CASES

In this section, we mention some special cases of our previous results as given below:

On putting, $\omega_2 = 0$ and replacing $\frac{\omega_1}{x}$ by z in (2.4), one obtains

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!}\ {}_2F_1\left[\begin{matrix}\lambda+n,\mu; \\ \alpha;\end{matrix} z\right] t^n = (1-t)^{-\lambda}\ {}_2F_1\left[\begin{matrix}\lambda,\mu; \\ \alpha;\end{matrix} \frac{z}{1-t}\right], \quad (3.1)$$

is the known result (see, [12], p.292, Eq.6).

Now, if on putting $\alpha_2 = \mu_2$, $\alpha_4 = \mu_4$, $\omega_1 = x_2\omega_1$, $\omega_2 = \frac{x_3 x_4 \omega_2}{x_2}$, in (2.7), one obtains the relation (2.4), which on further with usual replacement yield (3.1).

For $\omega_2 = 0$, in (2.11) and on replacing ω_1 by $\frac{z}{x}$, one obtains

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} F_1[1+\alpha,\lambda+n,\beta;1+\mu;z,x]t^n = (1-t)^{-\lambda} F_1\left[1+\alpha,\lambda,\beta;1+\mu;\frac{z}{1-t},x\right], \quad (3.2)$$

which for $x \to 0$ reduces to known result ([12], p.292, Eq.6).

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