On The Enestrom –Kakeya Theorem and Its Generalisations

[1,]M. H. Gulzar
[1,]Department of Mathematics University of Kashmir, Srinagar 190006

Abstract
Many extensions of the Enestrom –Kakeya Theorem are available in the literature. In this paper we prove some results which generalize some known results .

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1. Introduction And Statement Of Results

The Enestrom –Kakeya Theorem (see[7]) is well known in the theory of the distribution of zeros of polynomials and is often stated as follows:

Theorem A: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n whose coefficients satisfy

$$a_n \ge a_{n-1} \ge ...... \ge a_1 \ge a_0 > 0 .$$

Then P(z) has all its zeros in the closed unit disk $|z| \le 1$.

In the literature there exist several generalizations and extensions of this result. Joyal et al [6] extended it to polynomials with general monotonic coefficients and proved the following result:

Theorem B: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n whose coefficients satisfy

$$a_n \ge a_{n-1} \ge ...... \ge a_1 \ge a_0 .$$

Then P(z) has all its zeros in

$$|z| \le \frac{a_n - a_0 + |a_0|}{|a_n|} .$$

Aziz and zargar [1] generalized the result of Joyal et al [6] as follows:

Theorem C: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some $k \ge 1$

$$ka_n \ge a_{n-1} \ge ...... \ge a_1 \ge a_0 .$$

Then P(z) has all its zeros in

$$|z+k-1| \le \frac{ka_n - a_0 + |a_0|}{|a_n|} .$$

For polynomials ,whose coefficients are not necessarily real, Govil and Rahman [2] proved the following generalization of Theorem A:

Theorem C: If $P(z)=\sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n with $\mathrm{Re}(a_j)=\alpha_j$ and $\mathrm{Im}(a_j)=\beta_j$, j=0,1,2,……,n, such that

$$\alpha_n \ge \alpha_{n-1} \ge ...... \ge \alpha_1 \ge \alpha_0 \ge 0 ,$$

where $\alpha_n > 0$, then P(z) has all its zeros in

$$|z| \le 1 + (\frac{2}{\alpha_n})(\sum_{j=0}^{n}|\beta_j|) \ .$$

Govil and Mc-tume [3] proved the following generalisations of Theorems B and C:

Theorem D: Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $\text{Re}(a_j) = \alpha_j$ and $\text{Im}(a_j) = \beta_j$, j=0,1,2,……,n. If for some $k \ge 1$,

$$k\alpha_n \ge \alpha_{n-1} \ge ...... \ge \alpha_1 \ge \alpha_0 ,$$

then P(z) has all its zeros in

$$|z + k - 1| \le \frac{k\alpha_n - \alpha_0 + |\alpha_0| + 2\sum_{j=0}^{n}|\beta_j|}{|\alpha_n|} .$$

Theorem E: Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $\text{Re}(a_j) = \alpha_j$ and $\text{Im}(a_j) = \beta_j$, j=0,1,2,……,n. If for some $k \ge 1$,

$$k\beta_n \ge \beta_{n-1} \ge ...... \ge \beta_1 \ge \beta_0 ,$$

then P(z) has all its zeros in

$$|z + k - 1| \le \frac{k\beta_n - \beta_0 + |\beta_0| + 2\sum_{j=0}^{n}|\alpha_j|}{|\beta_n|} .$$

M. H. Gulzar [4] proved the following generalizations of Theorems D and E:

Theorem F: Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $\text{Re}(a_j) = \alpha_j$ and $\text{Im}(a_j) = \beta_j$, j=0,1,2,……,n. If for some real number $\rho \ge 0$,

$$\rho + \alpha_n \ge \alpha_{n-1} \ge ...... \ge \alpha_1 \ge \alpha_0 ,$$

then P(z) has all its zeros in the disk

$$\left|z + \frac{\rho}{\alpha_n}\right| \le \frac{\rho + \alpha_n + |\alpha_0| - \alpha_0 + 2\sum_{j=0}^{n}|\beta_j|}{|\alpha_n|} .$$

Theorem G: Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $\text{Re}(a_j) = \alpha_j$ and $\text{Im}(a_j) = \beta_j$, j=0,1,2,……,n. If for some real number $\rho \ge 0$,

$$\rho + \beta_n \ge \beta_{n-1} \ge ...... \ge \beta_1 \ge \beta_0 ,$$

then P(z) has all its zeros in the disk

$$\left|z + \frac{\rho}{\beta_n}\right| \le \frac{\rho + \beta_n + |\beta_0| - \beta_0 + 2\sum_{j=0}^{n}|\alpha_j|}{|\beta_n|} .$$

In this paper we give generalization of Theorems F and G. In fact, we prove the following:

Theorem 1: Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $\text{Re}(a_j) = \alpha_j$ and $\text{Im}(a_j) = \beta_j$, j=0,1,2,……,n. If for some real numbers $\lambda, \rho \ge 0$, $1 \le k \le n, \alpha_{n-k} \ne 0,$

$$\rho+\alpha_n \geq \alpha_{n-1} \geq ...\alpha_{n-k+1} \geq \lambda\alpha_{n-k} \geq \alpha_{n-k-1} \geq ... \geq \alpha_1 \geq \alpha_0,$$

and $\alpha_{n-k-1} > \alpha_{n-k}$, then P(z) has all its zeros in the disk

$$\left|z+\frac{\rho}{a_n}\right| \leq \frac{\rho+\alpha_n+(\lambda-1)\alpha_{n-k}+|\lambda-1||\alpha_{n-k}|+|\alpha_0|-\alpha_0+2\sum_{j=0}^{n}|\beta_j|}{|a_n|},$$

and if $\alpha_{n-k} > \alpha_{n-k+1}$, then P(z) has all its zeros in the disk

$$\left|z+\frac{\rho}{a_n}\right| \leq \frac{\rho+\alpha_n+(1-\lambda)\alpha_{n-k}+|1-\lambda||\alpha_{n-k}|+|\alpha_0|-\alpha_0+2\sum_{j=0}^{n}|\beta_j|}{|a_n|}$$

Remark 1: Taking $\lambda=1$, Theorem 1 reduces to Theorem F.
Taking k=n ,and $\lambda=1$ in Theorem 1, we get a result due to M.H. Gulzar [5, Theorem 1] .
If a_j are real i.e. $\beta_j=0$ for all j, we get the following result:

Corollary 1: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n. If for some real numbers $\lambda, \rho \geq 0$, $1 \leq k \leq n, a_{n-k} \neq 0,$

$$\rho+a_n \geq a_{n-1} \geq ... \geq a_{n-k+1} \geq \lambda a_{n-k} \geq a_{n-k-1} \geq ... \geq a_1 \geq a_0,$$

and $a_{n-k-1} > a_{n-k}$, then P(z) has all its zeros in the disk

$$\left|z+\frac{\rho}{a_n}\right| \leq \frac{\rho+a_n+(\lambda-1)a_{n-k}+|\lambda-1||a_{n-k}|+|a_0|-a_0}{|a_n|},,$$

and if $a_{n-k} > a_{n-k+1}$, then P(z) has all its zeros in the disk

$$\left|z+\frac{\rho}{a_n}\right| \leq \frac{\rho+a_n+(1-\lambda)a_{n-k}+|1-\lambda||a_{n-k}|+|a_0|-a_0}{|a_n|}$$

If we apply Theorem 1 to the polynomial –iP(z) , we easily get the following result:

Theorem 2: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $\text{Re}(a_j)=\alpha_j$ and $\text{Im}(a_j)=\beta_j$, j=0,1, 2,……,n. If for some real numbers $\lambda, \rho \geq 0$, $1 \leq k \leq n, \beta_{n-k} \neq 0,$

$$\rho+\beta_n \geq \beta_{n-1} \geq .... \geq \beta_{n-k+1} \geq \lambda\beta_{n-k} \geq \beta_{n-k-1} \geq .. \geq \beta_1 \geq \beta_0,$$

and $\beta_{n-k-1} > \beta_{n-k}$, then P(z) has all its zeros in the disk

$$\left|z+\frac{\rho}{a_n}\right| \leq \frac{\rho+\beta_n+(\lambda-1)\beta_{n-k}+|\lambda-1||\beta_{n-k}|+|\beta_0|-\beta_0+2\sum_{j=0}^{n}|\alpha_j|}{|a_n|},$$

and if $\beta_{n-k} > \beta_{n-k+1}$, then P(z) has all its zeros in the disk

$$\left|z+\frac{\rho}{a_n}\right| \leq \frac{\rho+\beta_n+(1-\lambda)\beta_{n-k}+|1-\lambda||\beta_{n-k}|+|\beta_0|-\beta_0+2\sum_{j=0}^{n}|\alpha_j|}{|a_n|}$$

Remark 2: Taking $\lambda=1$, Theorem 2 reduces to Theorem G.

Theorem 3: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some real numbers $\lambda, \rho \geq 0$, $1 \leq k \leq n, a_{n-k} \neq 0, and \beta$,

$$|\rho + a_n| \geq |a_{n-1}| \geq \ldots \geq |a_{n-k+1}| \geq \lambda |a_{n-k}| \geq |a_{n-k-1}| \geq \ldots \geq |a_1| \geq |a_0|$$

and

$$|\arg a_j - \beta| \leq \alpha \leq \frac{\pi}{2}, \; j = 0,1,\ldots\ldots,n.$$

If $|a_{n-k-1}| > |a_{n-k}|$ (i.e. $\lambda > 1$) , then P(z) has all its zeros in the disk

$$\left|z+\frac{\rho}{a_n}\right| \leq \frac{[|\rho + a_n|(\cos\alpha + \sin\alpha) - |a_{n-k}|(\cos\alpha - \sin\alpha - \lambda\cos\alpha - \lambda\sin\alpha - \lambda + 1) - |a_0|(\cos\alpha - \sin\alpha - 1) + 2\sin\alpha \sum_{j=1, j\neq n-k}^{n-1} |a_j|]}{|a_n|}$$

If $|a_{n-k}| > |a_{n-k+1}|$ (i.e. $\lambda < 1$), then P(z) has all its zeros in the disk

$$\left|z+\frac{\rho}{a_n}\right| \leq \frac{[|\rho + a_n|(\cos\alpha + \sin\alpha) + |a_{n-k}|(\cos\alpha + \sin\alpha - \lambda\cos\alpha + \lambda\sin\alpha + 1 - \lambda) + |a_0|(\sin\alpha - \cos\alpha + 1) + 2\sin\alpha \sum_{j=1, j\neq n-k}^{n-1} |a_j|]}{|a_n|}$$

Remark 3: Taking $\lambda = 1$ in Theorem 3 , we get the following result:

Corollary 2: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n. If for some real number $\rho \geq 0$,

$$|\rho + a_n| \geq |a_{n-1}| \geq \ldots\ldots \geq |a_1| \geq |a_0|,$$

then P(z) has all its zeros in the disk

$$\left|z+\frac{\rho}{a_n}\right| \leq \frac{[|\rho + a_n|(\cos\alpha + \sin\alpha) - |a_0|(\cos\alpha - \sin\alpha - 1) + 2\sin\alpha \sum_{j=1,}^{n-1} |a_j|]}{|a_n|}.$$

Remark 4: Taking $\rho = (k-1)a_n, k \geq 1$, we get a result of Shah and Liman [8,Theorem 1].

2. Lemmas

For the proofs of the above results , we need the following results:

Lemma 1: . Let P(z)=$\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients such that $|\arg a_j - \beta| \leq \alpha \leq \frac{\pi}{2}, j = 0,1,..n, for$ some real β. Then for some t>0,

$$|ta_j - a_{j-1}| \leq \big[t|a_j| - |a_{j-1}|\big]\cos\alpha + \big[t|a_j| + |a_{j-1}|\big]\sin\alpha.$$

The proof of lemma 1 follows from a lemma due to Govil and Rahman [2].

Lemma 2. If p(z) is regular ,p(0)$\neq$ 0 and $|p(z)| \leq M$ in $|z| \leq 1$, then the number of zeros of p(z)in $|z| \leq \delta, 0 < \delta < 1$, does not exceed $\frac{1}{\log\frac{1}{\delta}} \log \frac{M}{|p(0)|}$ (see[9],p171).

3. Proofs Of Theorems

Proof of Theorem 1: Consider the polynomial

$$F(z)=(1-z)P(z)$$
$$=(1-z)(a_n z^n+a_{n-1}z^{n-1}+\ldots\ldots+a_1 z+a_0)$$
$$=-a_n z^{n+1}+(a_n-a_{n-1})z^n+\ldots\ldots+(a_{n-k+1}-a_{n-k})z^{n-k+1}+(a_{n-k}-a_{n-k-1})z^{n-k}$$
$$+(a_{n-k-1}-a_{n-k-2})z^{n-k-1}+\ldots\ldots+(a_1-a_0)z+a_0$$
$$=-(\alpha_n+i\beta_n)z^{n+1}+(\alpha_n-\alpha_{n-1})z^n+\ldots\ldots+(\alpha_{n-k+1}-\alpha_{n-k})z^{n-k+1}$$
$$+(\alpha_{n-k}-\alpha_{n-k-1})z^{n-k}+(\alpha_{n-k-1}-\alpha_{n-k-2})z^{n-k-1}+\ldots\ldots+(\alpha_1-\alpha_0)z+\alpha_0$$
$$+i\sum_{j=1}^{n}(\beta_j-\beta_{j-1})z^j+i\beta_0$$

If $\alpha_{n-k-1}>\alpha_{n-k}$, then $\alpha_{n-k-1}>\alpha_{n-k}$, and we have

$$F(z)=-(\alpha_n+i\beta_n)z^{n+1}-\rho z^n+(\rho+\alpha_n-\alpha_{n-1})z^n+\ldots\ldots+(\alpha_{n-k+1}-\alpha_{n-k})z^{n-k+1}$$
$$+(\lambda\alpha_{n-k}-\alpha_{n-k-1})z^{n-k}-(\lambda-1)a_{n-k}z^{n-k}+(\alpha_{n-k-1}-\alpha_{n-k-2})z^{n-k-1}+\ldots\ldots$$
$$+(\alpha_1-\alpha_0)z+\alpha_0+i\sum_{j=1}^{n}(\beta_j-\beta_{j-1})z^j+i\beta_0.$$

For $|z|>1$,

$$|F(z)|\geq\left|a_n z^{n+1}+\rho z^n\right|-\Big|(\rho+\alpha_n-\alpha_{n-1})z^n+(\alpha_{n-1}-\alpha_{n-2})z^{n-1}+\ldots\ldots+(\alpha_{n-k+1}-\alpha_{n-k})z^{n-k+1}$$
$$+(\lambda\alpha_{n-k}-\alpha_{n-k-1})z^{n-k}+(\lambda-1)a_{n-k}z^{n-k}+(\alpha_{n-k-1}-\alpha_{n-k-2})z^{n-k-1}+\ldots\ldots$$
$$+(\alpha_1-\alpha_0)z+\alpha_0+i\sum_{j=1}^{n}(\beta_j-\beta_{j-1})z^j+i\beta_0\Big|$$
$$=|z|^n\Big[|a_n z+\rho|-\Big|(\rho+\alpha_n-\alpha_{n-1})+(\alpha_{n-1}-\alpha_{n-2})\frac{1}{z}+\ldots\ldots+(\alpha_{n-k+1}-\alpha_{n-k})\frac{1}{z^{k-1}}$$
$$+(\lambda\alpha_{n-k}-\alpha_{n-k-1})\frac{1}{z^k}-(\lambda-1)\alpha_{n-k}\frac{1}{z^k}+(\alpha_{n-k-1}-\alpha_{n-k-2})\frac{1}{k+1}+\ldots.$$
$$+(\alpha_1-\alpha_0)\frac{1}{z^{n-1}}+\frac{\alpha_0}{z^n}+i\sum_{j=1}^{n}(\beta_j-\beta_{j-1})\frac{1}{z^{n-j}}+i\frac{\beta_0}{z^n}\Big|\Big]$$
$$>|z|^n\Big[|a_n z+\rho|-\{|\rho+\alpha_n-\alpha_{n-1}|+|\alpha_{n-1}-\alpha_{n-2}|+\ldots\ldots+|\alpha_{n-k+1}-\alpha_{n-k}|$$
$$+|\lambda\alpha_{n-k}-\alpha_{n-k-1}|+|\lambda-1||\alpha_{n-k}|+|\alpha_{n-k-1}-\alpha_{n-k-2}|+\ldots\ldots+|\alpha_1-\alpha_0|$$
$$+|\alpha_0|+\sum_{j=1}^{n}|\beta_j-\beta_{j-1}|+|\beta_0|\}\Big]$$
$$\geq|z|^n\Big[|a_n z+\rho|-\{\rho+\alpha_n-\alpha_{n-1}+\alpha_{n-1}-\alpha_{n-2}+\ldots\ldots+\alpha_{n-k+1}-\alpha_{n-k}+\lambda\alpha_{n_{n-k}}-\alpha_{n-k-1}$$
$$+|\lambda-1||\alpha_{n-k}|+\alpha_{n-k-1}-\alpha_{n-k-2}+\ldots\ldots+\alpha_1-\alpha_0+|\alpha_0|+2\sum_{j=0}^{n}|\beta_j|\}\Big]$$
$$=|z|^n\Big[|a_n z+\rho|-\Big\{\rho+\alpha_n+(\lambda-1)\alpha_{n-k}+|\lambda-1||\alpha_{n-k}|-\alpha_0+|\alpha_0|+2\sum_{j=0}^{n}|\beta_j|\}\Big]$$
$$>0$$

if

$$\left|a_n z+\rho\right| > \rho+\alpha_n+(\lambda-1)\alpha_{n-k}+\left|\lambda-1\right|\left|\alpha_{n-k}\right|-\alpha_0+\left|\alpha_0\right|+2\sum_{j=0}^{n}\left|\beta_j\right|$$

This shows that the zeros of F(z) whose modulus is greater than 1 lie in

$$\left|z+\frac{\rho}{a_n}\right| \le \frac{\rho+\alpha_n+(\lambda-1)\alpha_{n-k}+\left|\lambda-1\right|\left|\alpha_{n-k}\right|+\left|\alpha_0\right|-\alpha_0+2\sum_{j=0}^{n}\left|\beta_j\right|}{\left|a_n\right|}.$$

But the zeros of F(z) whose modulus is less than or equal to 1 already satisfy the above inequality. Hence all the zeros of F(z) lie in

$$\left|z+\frac{\rho}{a_n}\right| \le \frac{\rho+\alpha_n+(\lambda-1)\alpha_{n-k}+\left|\lambda-1\right|\left|\alpha_{n-k}\right|+\left|\alpha_0\right|-\alpha_0+2\sum_{j=0}^{n}\left|\beta_j\right|}{\left|a_n\right|}.$$

Since the zeros of P(z) are also the zeros of F(z), it follows that all the zeros of P(z) lie in

$$\left|z+\frac{\rho}{a_n}\right| \le \frac{\rho+\alpha_n+(\lambda-1)\alpha_{n-k}+\left|\lambda-1\right|\left|\alpha_{n-k}\right|+\left|\alpha_0\right|-\alpha_0+2\sum_{j=0}^{n}\left|\beta_j\right|}{\left|a_n\right|}.$$

If $\alpha_{n-k}>\alpha_{n-k+1}$, then $\alpha_{n-k}>\alpha_{n-k-1}$, and we have

$$\begin{aligned}F(z)=&-(\alpha_n+i\beta_n)z^{n+1}-\rho z^n+(\rho+\alpha_n-\alpha_{n-1})z^n+......+(\alpha_{n-k+1}-\lambda\alpha_{n-k})z^{n-k+1}\\&+(\alpha_{n-k}-\alpha_{n-k-1})z^{n-k}-(1-\lambda)\alpha_{n-k}z^{n-k+1}+(\alpha_{n-k-1}-\alpha_{n-k-2})z^{n-k-1}+......\\&+(\alpha_1-\alpha_0)z+\alpha_0+i\sum_{j=1}^{n}(\beta_j-\beta_{j-1})z^j+i\beta_0.\end{aligned}$$

For $|z|>1$,

$$\begin{aligned}\left|F(z)\right|\ge&\left|a_n z^{n+1}+\rho z^n\right|-\Big|(\rho+\alpha_n-\alpha_{n-1})z^n+(\alpha_{n-1}-\alpha_{n-2})z^{n-1}+......+(\alpha_{n-k+1}-\lambda\alpha_{n-k})z^{n-k+1}\\&+(\alpha_{n-k}-\alpha_{n-k-1})z^{n-k}-(1-\lambda)\alpha_{n-k}z^{n-k+1}+(\alpha_{n-k-1}-\alpha_{n-k-2})z^{n-k-1}+......\\&+(\alpha_1-\alpha_0)z+\alpha_0+i\sum_{j=1}^{n}(\beta_j-\beta_{j-1})z^j+i\beta_0\Big|\\=&\left|z\right|^n\Bigg[\left|a_n z+\rho\right|-\Big|(\rho+\alpha_n-\alpha_{n-1})+(\alpha_{n-1}-\alpha_{n-2})\frac{1}{z}+......+(\alpha_{n-k+1}-\lambda\alpha_{n-k})\frac{1}{z^{k-1}}\\&+(\alpha_{n-k}-\alpha_{n-k-1})\frac{1}{z^k}-(1-\lambda)\alpha_{n-k}\frac{1}{z^{k-1}}+(\alpha_{n-k-1}-\alpha_{n-k-2})\frac{1}{z^{k+1}}+....\\&+(\alpha_1-\alpha_0)\frac{1}{z^{n-1}}+\frac{\alpha_0}{z^n}+i\sum_{j=1}^{n}(\beta_j-\beta_{j-1})\frac{1}{z^{n-j}}+i\frac{\beta_0}{z^n}\Big|\Bigg]\\>&\left|z\right|^n\Big[\left|a_n z+\rho\right|-\{\left|\rho+\alpha_n-\alpha_{n-1}\right|+\left|\alpha_{n-1}-\alpha_{n-2}\right|+......+\left|\alpha_{n-k+1}-\lambda\alpha_{n-k}\right|\\&+\left|\alpha_{n-k}-\alpha_{n-k-1}\right|+\left|1-\lambda\right|\left|\alpha_{n-k}\right|+\left|\alpha_{n-k-1}-\alpha_{n-k-2}\right|+......+\left|\alpha_1-\alpha_0\right|\\&+\left|\alpha_0\right|+\sum_{j=1}^{n}\left|\beta_j-\beta_{j-1}\right|+\left|\beta_0\right|\}\Big]\\\ge&\left|z\right|^n\Big[\left|a_n z+\rho\right|-\{\rho+\alpha_n-\alpha_{n-1}+\alpha_{n-1}-\alpha_{n-2}+......+\alpha_{n-k+1}-\lambda\alpha_{n-k}+\alpha_{n_{n-k}}-\alpha_{n-k-1}\end{aligned}$$

$$+|1-\lambda||\alpha_{n-k}|+\alpha_{n-k-1}-\alpha_{n-k-2}+......+\alpha_1-\alpha_0+|\alpha_0|+2\sum_{j=0}^{n}|\beta_j| \;\}\;]$$

$$=|z|^n\left[|a_nz+\rho|-\left\{\rho+\alpha_n+(1-\lambda)\alpha_{n-k}+|1-\lambda||\alpha_{n-k}|-\alpha_0+|\alpha_0|+2\sum_{j=0}^{n}|\beta_j| \;\}\;\right]\right.$$

>0

if

$$|a_nz+\rho|>\rho+\alpha_n+(1-\lambda)\alpha_{n-k}+|1-\lambda||\alpha_{n-k}|-\alpha_0+|\alpha_0|+2\sum_{j=0}^{n}|\beta_j|$$

This shows that the zeros of F(z) whose modulus is greater than 1 lie in

$$\left|z+\frac{\rho}{a_n}\right|\le\frac{\rho+\alpha_n+(1-\lambda)\alpha_{n-k}+|1-\lambda||\alpha_{n-k}|+|\alpha_0|-\alpha_0+2\sum_{j=0}^{n}|\beta_j|}{|a_n|}\;.$$

But the zeros of F(z) whose modulus is less than or equal to 1 already satisfy the above inequality. Hence all the zeros of F(z) lie in

$$\left|z+\frac{\rho}{a_n}\right|\le\frac{\rho+\alpha_n+(1-\lambda)\alpha_{n-k}+|1-\lambda||\alpha_{n-k}|+|\alpha_0|-\alpha_0+2\sum_{j=0}^{n}|\beta_j|}{|a_n|}\;.$$

Since the zeros of P(z) are also the zeros of F(z), it follows that all the zeros of P(z) lie in

$$\left|z+\frac{\rho}{a_n}\right|\le\frac{\rho+\alpha_n+(1-\lambda)\alpha_{n-k}+|1-\lambda||\alpha_{n-k}|+|\alpha_0|-\alpha_0+2\sum_{j=0}^{n}|\beta_j|}{|a_n|}\;.$$

That proves Theorem 1.

Proof of Theorem 3: Consider the polynomial

$$F(z)=(1-z)P(z)$$
$$=(1-z)(a_nz^n+a_{n-1}z^{n-1}+......+a_1z+a_0)$$
$$=-a_nz^{n+1}+(a_n-a_{n-1})z^n+......+(a_{n-k+1}-a_{n-k})z^{n-k+1}+(a_{n-k}-a_{n-k-1})z^{n-k}$$
$$+(a_{n-k-1}-a_{n-k-2})z^{n-k-1}+......+(a_1-a_0)z+a_0\;.$$

If $|a_{n-k-1}|>|a_{n-k}|$, then $|a_{n-k+1}|>|a_{n-k}|$, $\lambda>1$ and we have, for $|z|>1$, by using Lemma1,

$$|F(z)|\ge|a_nz^{n+1}+\rho z^n|-|(\rho+a_n-a_{n-1})z^n+(a_{n-1}-a_{n-2})z^{n-1}+......+(a_{n-k+1}-a_{n-k})z^{n-k+1}$$
$$+(\lambda a_{n-k}-a_{n-k-1})z^{n-k}+(\lambda-1)a_{n-k}z^{n-k}+(a_{n-k-1}-a_{n-k-2})z^{n-k-1}+......$$
$$+(a_1-a_0)z+a_0\;|$$
$$=|z|^n\Big[|a_nz+\rho|-|(\rho+a_n-a_{n-1})+(a_{n-1}-a_{n-2})\frac{1}{z}+......+(a_{n-k+1}-a_{n-k})\frac{1}{z^{k-1}}$$
$$+(\lambda a_{n-k}-a_{n-k-1})\frac{1}{z^k}+(\lambda-1)a_{n-k}\frac{1}{z^k}+(a_{n-k-1}-a_{n-k-2})\frac{1}{z^{k+1}}+....$$
$$+(a_1-a_0)\frac{1}{z^{n-1}}+\frac{a_0}{z^n}\;|\;\Big]$$

$$> |z|^n \big[|a_n z+\rho| - \{ \,|\rho + a_n - a_{n-1}| + |a_{n-1} - a_{n-2}| + \ldots\ldots + |a_{n-k+1} - a_{n-k}|$$
$$+ |\lambda a_{n-k} - a_{n-k-1}| + |\lambda - 1||a_{n-k}| + |a_{n-k-1} - a_{n-k-2}| + \ldots\ldots + |a_1 - a_0|$$
$$+ |a_0| \ \} \ \big]$$

$$\geq |z|^n \big[|a_n z+\rho| - \{ (|\rho + a_n| - |a_{n-1}|)\cos\alpha + (|\rho + a_n| + |a_{n-1}|)\sin\alpha$$
$$+ (|a_{n-1}| - |a_{n-2}|)\cos\alpha + (|a_{n-1}| + |a_{n-2}|)\sin\alpha + \ldots\ldots + (|a_{n-k+1}| - |a_{n-k}|)\cos\alpha$$
$$+ (\ |a_{n-k+1}| + |a_{n-k}|)\sin\alpha + (\lambda|a_{n-k}| - |a_{n-k-1}|)\cos\alpha + (\lambda|a_{n-k}| + |a_{n-k-1}|)\sin\alpha$$
$$+ |\lambda - 1||a_{n-k}| + (|a_{n-k-1}| - |a_{n-k-2}|)\cos\alpha + (|a_{n-k-1}| + |a_{n-k-2}|)\sin\alpha$$
$$+ \ldots\ldots + (|a_1| - |a_0|)\cos\alpha + (|a_1| + |a_0|)\sin\alpha + |a_0| \} \ \big]$$

$$= |z|^n \big[|a_n z+\rho| - \{ |\rho + a_n|(\cos\alpha + \sin\alpha) - |a_{n-k}|(\cos\alpha - \sin\alpha - \lambda\cos\alpha - \lambda\sin\alpha$$
$$- \lambda + 1) + |a_0|(\sin\alpha - \cos\alpha + 1) + 2\sin\alpha \sum_{j=1, j\neq n-k}^{n-1} |a_j| \} \ \big]$$

$$> 0$$

if

$$\big[|a_n z+\rho| > |\rho + a_n|(\cos\alpha + \sin\alpha) - |a_{n-k}|(\cos\alpha - \sin\alpha - \lambda\cos\alpha - \lambda\sin\alpha$$
$$- \lambda + 1) + |a_0|(\sin\alpha - \cos\alpha + 1) + 2\sin\alpha \sum_{j=1, j\neq n-k}^{n-1} |a_j|$$

This shows that the zeros of F(z) whose modulus is greater than 1 lie in

$$\left| z + \frac{\rho}{a_n} \right| \leq \frac{[|\rho + a_n|(\cos\alpha + \sin\alpha) - |a_{n-k}|(\cos\alpha - \sin\alpha - \lambda\cos\alpha - \lambda\sin\alpha - \lambda + 1) - |a_0|(\cos\alpha - \sin\alpha - 1) + 2\sin\alpha \sum_{j=1, j\neq n-k}^{n-1} |a_j|]}{|a_n|} \, ..$$

But the zeros of F(z) whose modulus is less than or equal to 1 already satisfy the above inequality. Hence all the zeros of F(z) lie in

$$\left| z + \frac{\rho}{a_n} \right| \leq \frac{[|\rho + a_n|(\cos\alpha + \sin\alpha) - |a_{n-k}|(\cos\alpha - \sin\alpha - \lambda\cos\alpha - \lambda\sin\alpha - \lambda + 1) - |a_0|(\cos\alpha - \sin\alpha - 1) + 2\sin\alpha \sum_{j=1, j\neq n-k}^{n-1} |a_j|]}{|a_n|} \, .$$

Since the zeros of P(z) are also the zeros of F(z), it follows that all the zeros of P(z) lie in

$$\left| z + \frac{\rho}{a_n} \right| \leq \frac{[|\rho + a_n|(\cos\alpha + \sin\alpha) - |a_{n-k}|(\cos\alpha - \sin\alpha - \lambda\cos\alpha - \lambda\sin\alpha - \lambda + 1) - |a_0|(\cos\alpha - \sin\alpha - 1) + 2\sin\alpha \sum_{j=1, j\neq n-k}^{n-1} |a_j|]}{|a_n|}$$

If $|a_{n-k}| > |a_{n-k+1}|$, then $|a_{n-k}| > |a_{n-k-1}|$, $\lambda < 1$ and we have, for $|z| > 1$, by using Lemma 1,

$$|F(z)| \geq |a_n z^{n+1} + \rho z^n| - |(\rho + a_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + \ldots\ldots + (a_{n-k+1} - \lambda a_{n-k})z^{n-k+1}$$
$$+ (a_{n-k} - a_{n-k-1})z^{n-k} - (1-\lambda)a_{n-k}z^{n-k+1} + (a_{n-k-1} - a_{n-k-2})z^{n-k-1} + \ldots\ldots$$
$$+ (a_1 - a_0)z + a_0 \ \ |$$

$$= |z|^n \Big[|a_n z + \rho| - \Big| (\rho + a_n - a_{n-1}) + (a_{n-1} - a_{n-2})\frac{1}{z} + \ldots\ldots + (a_{n-k+1} - \lambda a_{n-k})\frac{1}{z^{k-1}}$$

$$+ (a_{n-k} - a_{n-k-1})\frac{1}{z^k} - (1-\lambda)a_{n-k}\frac{1}{z^{k-1}} + (a_{n-k-1} - a_{n-k-2})\frac{1}{z^{k+1}} + \ldots.$$

$$+ (a_1 - a_0)\frac{1}{z^{n-1}} + \frac{a_0}{z^n} \Big| \Big]$$

$$> |z|^n \big[|a_n z + \rho| - \{ |\rho + a_n - a_{n-1}| + |a_{n-1} - a_{n-2}| + \ldots\ldots + |a_{n-k+1} - \lambda a_{n-k}|$$

$$+ |a_{n-k} - a_{n-k-1}| + |1-\lambda||a_{n-k}| + |a_{n-k-1} - a_{n-k-2}| + \ldots\ldots + |a_1 - a_0|$$

$$+ |a_0| \} \big]$$

$$\geq |z|^n \big[|a_n z + \rho| - \{ (|\rho + a_n| - |a_{n-1}|)\cos\alpha + (|\rho + a_n| + |a_{n-1}|)\sin\alpha$$

$$+ (|a_{n-1}| - |a_{n-2}|)\cos\alpha + (|a_{n-1}| + |a_{n-2}|)\sin\alpha + \ldots\ldots + (|a_{n-k+1}| - |\lambda a_{n-k}|)\cos\alpha$$

$$+ (|a_{n-k+1}| + |\lambda a_{n-k}|)\sin\alpha + (|a_{n-k}| - |a_{n-k-1}|)\cos\alpha + (|a_{n-k}| + |a_{n-k-1}|)\sin\alpha$$

$$+ |1-\lambda||a_{n-k}| + (|a_{n-k-1}| - |a_{n-k-2}|)\cos\alpha + (|a_{n-k-1}| + |a_{n-k-2}|)\sin\alpha$$

$$+ \ldots\ldots + (|a_1| - |a_0|)\cos\alpha + (|a_1| + |a_0|)\sin\alpha + |a_0| \} \big]$$

$$= |z|^n \big[|a_n z + \rho| - \{ |\rho + a_n|(\cos\alpha + \sin\alpha) + |a_{n-k}|(\cos\alpha + \sin\alpha - \lambda\cos\alpha + \lambda\sin\alpha$$

$$+ 1 - \lambda) + |a_0|(\sin\alpha - \cos\alpha + 1) + 2\sin\alpha \sum_{j=1, j\neq n-k}^{n-1} |a_j| \} \big]$$

$$> 0$$

if

$$\big[|a_n z + \rho| > |\rho + a_n|(\cos\alpha + \sin\alpha) + |a_{n-k}|(\cos\alpha + \sin\alpha - \lambda\cos\alpha + \lambda\sin\alpha$$

$$+ 1 - \lambda) + |a_0|(\sin\alpha - \cos\alpha + 1) + 2\sin\alpha \sum_{j=1, j\neq n-k}^{n-1} |a_j|$$

This shows that the zeros of F(z) whose modulus is greater than 1 lie in

$$\left| z + \frac{\rho}{a_n} \right| \leq \frac{[|\rho + a_n|(\cos\alpha + \sin\alpha) + |a_{n-k}|(\cos\alpha + \sin\alpha - \lambda\cos\alpha + \lambda\sin\alpha + 1 - \lambda) + |a_0|(\sin\alpha - \cos\alpha + 1) + 2\sin\alpha \sum_{j=1, j\neq n-k}^{n-1} |a_j|]}{|a_n|}.$$

But the zeros of F(z) whose modulus is less than or equal to 1 already satisfy the above inequality. Hence all the zeros of F(z) and therefore P(z) lie in

$$\left| z + \frac{\rho}{a_n} \right| \leq \frac{[|\rho + a_n|(\cos\alpha + \sin\alpha) + |a_{n-k}|(\cos\alpha + \sin\alpha - \lambda\cos\alpha + \lambda\sin\alpha + 1 - \lambda) + |a_0|(\sin\alpha - \cos\alpha + 1) + 2\sin\alpha \sum_{j=1, j\neq n-k}^{n-1} |a_j|]}{|a_n|}$$

That proves Theorem 3.

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