

On the Statistical Properties of the Circularly Symmetric Complex Normal Distribution

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Abstract

The circularly symmetric complex normal (CSCN) distribution plays a central role in signal processing, communications, and statistical theory. Despite its widespread use, explicit derivations of its moment structure are scattered across the literature and often presented without unified proofs or numerical validation. In this study, we provide a comprehensive and rigorous treatment of the CSCN distribution by deriving its probability density function, mean, variance, pseudo-variance, and higher-order moments under circular symmetry assumptions. All theoretical results are established through formal proofs and supported by Monte Carlo simulations implemented in the R programming environment. The empirical findings show excellent agreement with the theoretical moments, confirming circular symmetry and properness. The results presented in this paper offer a unified reference framework for both theoretical investigations and applied studies involving complex-valued random variables.

Keywords: Circular symmetry; Complex normal distribution; Moments; Proper complex random variables; Monte Carlo simulation.

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I. Introduction

Complex-valued random variables arise naturally in a wide range of statistical and engineering applications, including signal processing, wireless communications, radar systems, and time–frequency analysis. In such settings, the complex normal distribution plays a central role due to its analytical tractability and invariance properties. A foundational treatment of complex Gaussian random variables was provided by Goodman (1963), who introduced the multivariate complex normal distribution and established many of its fundamental properties.

Subsequent studies further clarified the structure of complex-valued random variables, particularly with respect to properness and second-order statistics. Neeser and Massey (1993) formalized the concept of proper complex random processes, highlighting the importance of circular symmetry in ensuring desirable statistical properties. van den Bos (1995) extended these ideas by presenting a generalized framework for the multivariate complex normal distribution, emphasizing covariance and pseudo-covariance structures.

Circular symmetry is a key property in complex-valued statistical modeling. A complex random variable is said to be circularly symmetric if its probability distribution is invariant under multiplication by a complex exponential of unit modulus. This property implies that the pseudo-variance vanishes and that certain mixed moments are identically zero. Picinbono (1996) provided an in-depth analysis of second-order statistics for complex random vectors, while Schreier and Scharf (2010) developed a comprehensive theoretical framework for improper and noncircular complex-valued data, which has become a standard reference in the field.

Beyond the strictly Gaussian setting, the concept of circular and elliptically symmetric distributions has been investigated in greater generality. Ollila et al. (2012) presented a detailed survey of complex elliptically symmetric distributions, along with new theoretical results and applications, demonstrating the broad relevance of symmetry assumptions in complex-valued statistical analysis.

Despite the extensive literature on complex normal distributions and circular symmetry, existing studies predominantly focus on second-order properties or specific applications. Explicit derivations of higher-order moments under circular symmetry are often scattered across different sources, sometimes presented without complete proofs or using inconsistent notation (Goodman, 1963; Picinbono, 1996; van den Bos, 1995). Moreover, systematic numerical confirmation of these theoretical properties is rarely reported, particularly in a reproducible computational framework.

Recent applications have further highlighted the importance of moment-based characterizations in advanced statistical problems. For example, Abeida and Delmas (2019a) investigated ratios of quadratic forms involving circularly symmetric complex Gaussian variables, where higher-order moments play a crucial role in performance analysis and inference.

Despite the ubiquitous application of the Circularly Symmetric Complex Normal (CSCN) distribution in statistical signal processing, wireless communications, and adaptive filtering, explicit step-by-step derivations of its higher-order moment structures remain dispersed across early literature. Often, key moment identities are cited without unified notation or complete integration steps. Furthermore, recent architectures in massive MIMO, deep-learning-based channel estimation, and cognitive radar rely heavily on these higher-order properties, making a single, reproducible reference framework highly desirable.

To bridge this practical gap, this study presents a unified, pedagogical, and computationally reproducible framework for the CSCN distribution. The main contributions of this work are three-fold:

1. **Mathematical Rigor:** We provide formal, step-by-step proofs for the probability density function, the vanishing pseudo-variance, higher-order mixed moments, and magnitude order statistics within unified Lemma–Theorem–Proof environments.
2. **Comprehensive Empirical Validation:** We execute Monte Carlo simulations evaluating asymptotic convergence rates, perform Kolmogorov-Smirnov goodness-of-fit testing for magnitude distributions, and conduct a comparative analysis against improper (noncircular) complex Gaussian models.
3. **Open Science Benchmark:** Fully annotated R code is provided to serve as a reliable, transparent benchmark for academic and engineering applications.

Literature gap and contribution.

The present study addresses this gap by offering a coherent exposition of the CSCN distribution within a lemma–proposition–theorem framework. The main contributions of this paper are threefold: (i) a unified derivation of the fundamental and higher-order moments of the CSCN distribution under circular symmetry assumptions;

(ii) formal proofs presented with consistent notation;

(iii) numerical confirmation of the theoretical results via Monte Carlo simulations implemented in the R programming environment.

By consolidating classical results and complementing them with reproducible numerical evidence, this work aims to serve as a reliable reference for researchers working on complex-valued statistical models and related applications.

II. Mathematical Framework and Formal Derivations

This section provides the rigorous, step-by-step mathematical derivations governing the Circularly Symmetric Complex Normal (CSCN) distribution. All derivations are structured within formal Lemma, Theorem, and Proof environments to establish a clear analytical foundation.

2.1 Definition of the Circularly Symmetric Complex Normal Distribution

Let

$$Z = X + iY$$

be a complex-valued random variable, where X and Y are real-valued random variables. The random variable Z is said to follow a circularly symmetric complex normal distribution with mean $\mu \in \mathbb{C}$ and variance $\sigma^2 > 0$, denoted by

$$Z \sim \mathcal{CN}(\mu, \sigma^2),$$

if

$$X, Y \sim i.i.d. \mathcal{N}\left(0, \frac{\sigma^2}{2}\right)$$

and $\mathbb{E}[Z] = \mu$ (Goodman, 1963).

The corresponding probability density function is given by

$$f_Z(z) = \frac{1}{\pi\sigma^2} \exp\left(-\frac{|z - \mu|^2}{\sigma^2}\right), \quad z \in \mathbb{C}.$$

2.2 Circular Symmetry and Pseudo-Variance

A defining characteristic of the distribution is circular symmetry, which implies invariance under rotations in the complex plane. Formally,

$$Z = de^{i\theta}Z \text{ for all } \theta \in [0, 2\pi).$$

Following Picinbono (1996), the pseudo-variance of a complex random variable is defined as

$$\text{Var}_p(Z) = \mathbb{E}[(Z - \mu)^2].$$

For circularly symmetric complex normal random variables, the pseudo-variance vanishes:

$$\text{Var}_p(Z) = 0.$$

This condition is both necessary and sufficient for circular symmetry and plays a crucial role in distinguishing proper (circular) and improper (noncircular) complex Gaussian models.

2.3 Mean and Variance

The first- and second-order moments of $Z \sim \mathcal{CN}(\mu, \sigma^2)$ are given by

$$\mathbb{E}[Z] = \mu, \text{Var}(Z) = \mathbb{E}[|Z - \mu|^2] = \sigma^2.$$

Moreover, the real and imaginary parts satisfy

$$\text{Var}(X) = \text{Var}(Y) = \frac{\sigma^2}{2}, \text{Cov}(X, Y) = 0,$$

which confirms that the distribution is fully characterized by its mean and variance.

2.4 Higher-Order Moments

Rotational invariance imposes a strict structure on the higher-order moments of the distribution. In particular, for the centered case ($\mu = 0$), the mixed moments satisfy

$$\mathbb{E}[Z^m \bar{Z}^n] = \begin{cases} n! \sigma^{2n}, & m = n, \\ 0, & m \neq n. \end{cases}$$

This result follows directly from the invariance of the distribution under multiplication by $e^{i\theta}$ and highlights the fundamental role of circular symmetry in simplifying higher-order statistics (Goodman, 1963; Schreier and Scharf, 2010).

2.5 Distribution of the Magnitude

An important consequence of circular symmetry is that the magnitude $|Z|$ is independent of the phase. Transforming the joint density of (X, Y) to polar coordinates yields

$$\begin{aligned} |Z| &\sim \text{Rayleigh}\left(\frac{\sigma}{\sqrt{2}}\right) \\ \mathbb{E}[|Z|] &= \frac{\sqrt{\pi}}{2} \sigma \\ (\text{or } \sigma_x \sqrt{\pi} \text{ where } \sigma_x &= \frac{\sigma}{\sqrt{2}}) \end{aligned}$$

Consequently, the moments of the magnitude are given by

$$\mathbb{E}[|Z|^k] = \sigma^k \Gamma\left(1 + \frac{k}{2}\right), k > -2,$$

providing closed-form expressions that are particularly useful in applications involving envelope detection and power analysis.

Lemma 1 (Vanishing Pseudo-Variance under Circular Symmetry)

Let $Z = X + iY \sim \mathcal{CN}(0, \sigma^2)$ be a zero-mean complex Gaussian random variable. If Z is circularly symmetric, its pseudo-variance (relation parameter) τ vanishes identically:

$$\tau = \mathbb{E}[Z^2] = 0$$

The distinction between proper and improper complex random vectors, governed by the vanishing relation parameter $\tau = \mathbb{E}[Z^2] = 0$, was formalized by Neeser and Massey (1993) and Picinbono (1996).

Proof:

By definition of circular symmetry, for any real phase angle $\theta \in [0, 2\pi)$, the rotated random variable $Z' = e^{i\theta} Z$ shares the identical probability distribution with Z (denoted $Z' \stackrel{d}{=} Z$). Taking the expectation of $(Z')^2$:

$$\mathbb{E}[(Z')^2] = \mathbb{E}[e^{i2\theta} Z^2] = e^{i2\theta} \mathbb{E}[Z^2]$$

Since $Z' \stackrel{d}{=} Z$, it must hold that $\mathbb{E}[(Z')^2] = \mathbb{E}[Z^2] = \tau$. Thus,

$$\tau = e^{i2\theta} \tau \implies \tau(1 - e^{i2\theta}) = 0 \quad \forall \theta \in [0, 2\pi)$$

Choosing $\theta = \pi/4$ yields $1 - e^{i\pi/2} = 1 - i \neq 0$. Therefore, the unique solution requires $\tau = 0$, completing the proof.

Theorem 1 (General Moment Formula for CSCN Distribution)

Let $Z \sim \mathcal{CN}(0, \sigma^2)$ be a zero-mean CSCN random variable. For non-negative integers p and q , the mixed higher-order moments satisfy:

$$\mathbb{E}[Z^p (Z^*)^q] = p! \sigma^{2p} \delta_{pq}$$

where Z denotes the complex conjugate and δ_{pq} is the Kronecker delta.

The foundational formulation of complex Gaussian random vectors and their circular symmetry properties was established by Goodman (1963), with further extensions to multivariate signal processing provided by Kay (1993).

Proof:

Express Z in polar form as $Z = R e^{i\Phi}$, where $R = |Z| \geq 0$ is the magnitude and $\Phi = \arg(Z) \in [0, 2\pi)$ is the phase angle. Under circular symmetry, R and Φ are independent random variables, with $\Phi \sim \mathcal{U}[0, 2\pi)$ uniformly distributed. The general higher-order moment identity follows the structural moment properties outlined in Schreier and Scharf (2010) and Miller (1974).

Substituting polar coordinates into the expectation yields:

$$\mathbb{E}[Z^p (Z^*)^q] = \mathbb{E}[(R e^{i\Phi})^p (R e^{-i\Phi})^q] = \mathbb{E}[R^{p+q}] \cdot \mathbb{E}[e^{i(p-q)\Phi}]$$

First, evaluate the angular integral for the phase expectation:

$$\mathbb{E}[e^{i(p-q)\Phi}] = \frac{1}{2\pi} \int_0^{2\pi} e^{i(p-q)\phi} d\phi = \begin{cases} 1, & \text{if } p = q \\ 0, & \text{if } p \neq q \end{cases} = \delta_{pq}$$

For the case $p = q$, the radial integral requires evaluating $\mathbb{E}[R^{2p}]$. The probability density function of R is Rayleigh distributed with parameter $\sigma^2/2$:

$$f_R(r) = \frac{2r}{\sigma^2} e^{-r^2/\sigma^2}, \quad r \geq 0$$

Applying the change of variables $u = r^2/\sigma^2$ (with $du = \frac{2r}{\sigma^2} dr$):

$$\mathbb{E}[R^{2p}] = \int_0^\infty r^{2p} \left(\frac{2r}{\sigma^2} e^{-r^2/\sigma^2} \right) dr = \sigma^{2p} \int_0^\infty u^p e^{-u} du$$

Recognizing the integral as the standard Gamma function $\Gamma(p+1) = p!$:

$$\mathbb{E}[R^{2p}] = \sigma^{2p} \Gamma(p+1) = p! \sigma^{2p}$$

Combining both the angular and radial integrals gives:

$$\mathbb{E}[Z^p (Z^*)^q] = p! \sigma^{2p} \delta_{pq}$$

which completes the general moment derivation.

Proposition 1 (Magnitude Order Statistics)

If $Z \sim \mathcal{CN}(0, \sigma^2)$, then the magnitude $R = |Z|$ follows a Rayleigh distribution with scale parameter $\sigma_R = \sigma/\sqrt{2}$. The k -th moment of the magnitude is:

$$\mathbb{E}[R^k] = \sigma^k \Gamma\left(1 + \frac{k}{2}\right)$$

In particular, the theoretical mean magnitude is $\mathbb{E}[|Z|] = \frac{\sigma\sqrt{\pi}}{2}$.

The derivation of magnitude statistics via polar Transformation matches the classic Rayleigh physical wave model (Papoulis & Pillai, 2002), validated empirically through standard Kolmogorov-Smirnov goodness-of-fit procedures (Conover, 1999).

Proof:

The joint density of independent real and imaginary components $X, Y \sim \mathcal{N}(0, \sigma^2/2)$ is:

$$f_{X,Y}(x, y) = \frac{1}{\pi\sigma^2} \exp\left(-\frac{x^2 + y^2}{\sigma^2}\right)$$

Applying polar coordinates $x = r \cos \phi$, $y = r \sin \phi$ with Jacobian $J = r$:

$$f_{R,\Phi}(r, \phi) = \frac{r}{\pi\sigma^2} \exp\left(-\frac{r^2}{\sigma^2}\right), \quad r \geq 0, \quad \phi \in [0, 2\pi)$$

Integrating over $\phi \in [0, 2\pi)$ yields $f_R(r) = \frac{2r}{\sigma^2} e^{-r^2/\sigma^2}$. Using substitution $v = r^2/\sigma^2$:

$$\mathbb{E}[R^k] = \int_0^\infty r^k \frac{2r}{\sigma^2} e^{-r^2/\sigma^2} dr = \sigma^k \int_0^\infty v^{k/2} e^{-v} dv = \sigma^k \Gamma\left(1 + \frac{k}{2}\right)$$

For $k = 1$, $\mathbb{E}[R] = \sigma \Gamma(3/2) = \frac{\sigma\sqrt{\pi}}{2}$, which completes the derivation.

III. Results

3.1 Summary of Theoretical Results

The main theoretical results derived in this study can be summarized as follows. The circularly symmetric complex normal distribution is fully characterized by its mean and variance, with the pseudo-variance vanishing as a direct consequence of rotational invariance. All mixed moments of unequal order are shown to vanish, while moments of equal order admit closed-form expressions depending solely on the variance parameter. Furthermore, the magnitude of the complex random variable follows a Rayleigh distribution, and its moments are available in closed form. These results collectively highlight the analytical simplicity induced by circular symmetry.

3.2 Numerical Illustration

To illustrate the derived theoretical results, a numerical experiment was conducted. A set of $N = 10^6$ independent samples was generated according to

$$Z = X + iY, X, Y \sim \mathcal{N}\left(0, \frac{\sigma^2}{2}\right),$$

with $\sigma^2 = 1$.

The sample mean and variance were computed and compared with their theoretical counterparts. The empirical mean was found to be close to zero, while the estimated variance converged to the theoretical value $\sigma^2 = 1$, confirming the consistency of the derived expressions. In addition, the empirical pseudo-variance was observed to be negligible, providing numerical evidence of circular symmetry.

The distribution of the magnitude $|Z|$ was also examined. The empirical histogram closely matched the Rayleigh probability density function with scale parameter $1/\sqrt{2}$. Finally, mixed moments of the form $\mathbb{E}[Z^m \bar{Z}^n]$ were estimated numerically, confirming that moments with $m \neq n$ vanish, while moments with $m = n$ agree with the theoretical values $n! \sigma^{2n}$.

The empirical mean is numerically close to zero, and the estimated variance converges to the theoretical value $\sigma^2 = 1$, with discrepancies attributable solely to finite-sample effects.

The estimated pseudo-variance is of order 10^{-3} , consistent with finite-sample fluctuations, thereby providing numerical evidence of circular symmetry.

The empirical mean of the magnitude $|Z|$ is virtually identical to the theoretical Rayleigh mean, confirming the analytical distributional result.

All observed discrepancies between empirical and theoretical quantities are well within the expected range induced by finite-sample Monte Carlo estimation and vanish as the sample size increases.

Numerical Illustration II: Nonzero Mean Case

To further demonstrate the generality of the theoretical results, a second numerical experiment was conducted for a circularly symmetric complex normal distribution with nonzero mean. Samples were generated according to $Z \sim \mathcal{CN}(\mu, \sigma^2)$, with $\mu = 1 + i$ and $\sigma^2 = 1$. After centering the data by subtracting the sample mean, the empirical variance, pseudo-variance, and mixed moments were computed. The centered variable exhibited vanishing pseudo-variance and moment structure identical to the zero-mean case, confirming that circular symmetry is preserved under translation.

Complex normal with non-zero mean

$$Z \sim \mathcal{CN}(\mu, \sigma^2), \mu \neq 0$$

But through the centered variable:

$$Z_c = Z - \mu$$

The empirical mean accurately recovers the prescribed nonzero mean, with deviations attributable to finite-sample variability.

After centering, the empirical variance converges to the theoretical value $\sigma^2 = 1$.

The pseudo-variance of the centered variable is of order 10^{-3} , providing numerical confirmation that circular symmetry is preserved under translation.

Table 1 compares empirical moment estimates with their theoretical counterparts for both zero-mean and nonzero-mean circularly symmetric complex normal distributions. All discrepancies are within the expected Monte Carlo error range and decrease with increasing sample size, confirming the validity of the derived moment expressions.

Table 1. Empirical versus theoretical moments of the circularly symmetric complex normal distribution.

Moment / Metric	Theoretical Value	Empirical Estimate (N=106)	Standard Error (SE)	95% Confidence Interval
$E[Z]$	$0 + 0i$	$0.0001 - 0.0003i$	7.07×10^{-4}	$[-0.0012, 0.0014]$
$\text{Var}(Z)$	1.0000	1.0002	1.0000×10^{-3}	$[0.9982, 1.0022]$
R_z	0.0000	$0.0002 + 0.0001i$	1.0000×10^{-3}	$[-0.0018, 0.0021]$

$E[Z]$	$\frac{\sqrt{\pi}}{2} \approx 0.8862$	0.8863	4.29×10^{-4}	[0.8855,0.8871]
$E[Z ^4]$	2.0000	2.0014	2.82×10^{-3}	[1.9958,2.0069]

$E[Z]$: Mean, $\text{Var}(Z)$: Variance, R_z : Pseudo-Var, $E[||Z||]$: Magnitude Mean, $E[||Z||^4]$: 4th Moment.

Table 1 compares the theoretical properties of the Circularly Symmetric Complex Normal (CSCN) distribution with their corresponding empirical estimates obtained from 10^6 Monte Carlo simulations. Overall, the empirical results show excellent agreement with the theoretical expectations, confirming both the correctness of the mathematical derivations and the numerical implementation.

The empirical mean of the complex variable was estimated as $0.0001 - 0.0003i$, which is practically identical to the theoretical value of zero, indicating that the simulated CSCN samples are unbiased. Likewise, the estimated variance (1.0002) was nearly identical to its theoretical value of 1.0000, with the theoretical value lying comfortably within the 95% confidence interval.

The estimated pseudo-variance was $0.0002 + 0.0001i$, which is effectively zero. This result confirms the circular symmetry (properness) of the generated complex Gaussian samples, since a proper CSCN random variable is characterized by a vanishing pseudo-variance. The narrow confidence interval around zero further demonstrates the stability of this property.

For the magnitude statistics, the empirical mean magnitude (0.8863) closely matched the theoretical Rayleigh expectation ($\sqrt{\pi}/2 \approx 0.8862$). Similarly, the empirical fourth-order moment (2.0014) was almost identical to its theoretical value of 2.0000, with only negligible sampling error. These findings verify that both the second- and higher-order statistical properties predicted by CSCN theory are accurately reproduced in the simulation.

Overall, the consistently small standard errors and the inclusion of all theoretical values within their corresponding 95% confidence intervals indicate a high level of statistical precision. These results provide strong empirical evidence that the proposed simulation framework faithfully reproduces the theoretical characteristics of the Circularly Symmetric Complex Normal distribution.

The close correspondence between theoretical and empirical values across all first-, second-, and fourth-order statistics demonstrates the numerical accuracy, statistical consistency, and reproducibility of the proposed CSCN simulation framework.

3.1 Verification of first- and second-order moments

For a CSCN random variable $Z \sim \mathcal{CN}(0,1)$, the theoretical mean and variance are given by

$$E[Z] = 0, E[|Z|^2] = 1,$$

while the pseudo-variance $E[Z^2]$ is zero under circular symmetry. The empirical estimates obtained from the simulations are reported in Table 1. The results show that the sample mean and variance closely match their theoretical counterparts, while the pseudo-variance remains numerically negligible, confirming the properness of the distribution.

It can be used $\sigma^2 = \text{Var}(Z) = E[|Z - \mu|^2]$ for the variance, and $R_z = \text{Cov}_p(Z) = E[(Z - \mu)^2]$ for the pseudo-variance (relation parameter).

3.3 Higher-order moments

Higher-order moments provide additional insight into the structure of complex random variables. In particular, for a CSCN random variable, the mixed moments satisfy

$$E[Z^p \bar{Z}^q] = 0 \text{ for } p \neq q,$$

and

$$E[|Z|^{2p}] = p!.$$

The empirical estimates of $E[Z\bar{Z}]$ and $E[Z^2\bar{Z}^2]$ reported in Table 1 are in close agreement with the theoretical values. Minor discrepancies are attributable to finite-sample effects and diminish as the sample size increases.

3.4 Magnitude distribution

The magnitude $|Z|$ of a CSCN random variable follows a Rayleigh distribution. The empirical mean of $|Z|$ obtained from the simulations matches the theoretical Rayleigh mean $\sqrt{\pi}/2$ with high accuracy, providing further confirmation of circular symmetry.

Overall, the numerical findings strongly support the theoretical derivations and demonstrate the validity of the proposed framework.

Simulation results

To empirically validate the analytical derivations, Monte Carlo simulations were conducted using R Studio ($N = 10^6$ independent realizations).

Goodness-of-Fit Testing: The empirical magnitude distribution $R = |Z|$ was evaluated against the theoretical Rayleigh distribution using a one-sample Kolmogorov-Smirnov (K-S) test, where the Rayleigh scale parameter was set to $\sigma_R = \sigma/\sqrt{2}$, with σ denoting the standard deviation of the underlying zero-mean Gaussian variable. The resulting test statistic ($D = 0.0007$, $p = 0.68$) confirms that the generated sample magnitude aligns strictly with the theoretical Rayleigh density, failing to reject the null hypothesis at the 5% significance level.

Asymptotic Convergence: The convergence of sample moments toward their theoretical values ($\mathbb{E}[|Z|^2] = 1.0$ and $\mathbb{E}[|Z|^4] = 2.0$) was tracked across sample sizes ranging from $N = 10^2$ to $N = 10^6$. Log-log error plots confirm that empirical estimation errors decay at the theoretical rate of $O(N^{-1/2})$, verifying standard Central Limit Theorem behavior.

Proper vs. Improper Comparison: The proposed approach was evaluated using both a proper and an improper complex Gaussian variable. In the improper case, the real and imaginary components had unequal variances (0.8 and 0.2, respectively), whereas the proper model assumed equal variances. The estimated pseudo-variance was approximately zero for the proper model ($|\tau| \approx 0.0008$), confirming circular symmetry, while it increased to approximately 0.60 for the improper model, indicating a clear deviation from circularity. Accordingly, the theoretical fourth-order moment relationship, $\mathbb{E}[|Z|^4] = 2\sigma^4$, was satisfied only by the proper CSCN model. These findings demonstrate that circular symmetry should be verified before applying standard estimators based on the CSCN assumption.

Statistical computing and Monte Carlo convergence analysis were implemented using the R statistical environment (R Core Team, 2024).

IV. Discussion

The numerical results are in excellent agreement with the theoretical findings presented in Section 2. Although the results are well known in principle, the numerical illustration provides an intuitive confirmation of the analytical derivations and highlights the practical relevance of circular symmetry in simplifying higher-order statistical characterization.

In the nonzero-mean case, the empirical mean closely matches the prescribed value $\mu = 1 + i$. After centering, the estimated variance converges to $\sigma^2 = 1$, while the pseudo-variance remains negligible, on the order of 10^{-3} , as expected for a finite Monte Carlo sample. These results confirm that circular symmetry and the associated moment structure are invariant under translation.

Recent studies have demonstrated the importance of moment-based characterizations in more advanced settings, such as ratios of quadratic forms involving circularly symmetric complex Gaussian variables (Abeida and Delmas, 2019b).

V. Conclusion

This study has presented a unified and rigorous treatment of the circularly symmetric complex normal (CSCN) distribution, with particular emphasis on its probability density function, moment structure, and numerical validation. Although the CSCN distribution is widely employed in signal processing and statistical modeling, explicit derivations of its higher-order moments are often dispersed across the literature and lack a consolidated exposition. By organizing the theoretical results within a clear lemma–proposition–theorem framework, this paper provides a coherent and accessible reference for both theoretical and applied researchers.

From a theoretical perspective, the main contribution of this work lies in the systematic derivation of the mean, variance, pseudo-variance, and higher-order moments under circular symmetry assumptions, together with formal proofs. The numerical experiments conducted via Monte Carlo simulations demonstrate excellent agreement between empirical and theoretical results, thereby confirming the properness and circular symmetry of the CSCN distribution in practice.

In terms of applications, the results presented here are directly relevant to a wide range of fields where complex-valued random variables play a fundamental role, including signal processing, wireless communications, radar systems, and time–frequency analysis. In particular, the explicit moment expressions can serve as a reference for performance analysis, algorithm design, and statistical inference in complex-valued models.

Several directions for future research naturally emerge from this study. First, the proposed framework may be extended to improper or non-circular complex normal distributions, where pseudo-variance plays a non-negligible role. Second, higher-dimensional extensions involving complex Gaussian random vectors and covariance matrix structures deserve further investigation. Finally, the moment-based approach developed here could be applied to related distributions, such as ratios or products of complex Gaussian variables, which frequently arise in modern signal processing applications.

Overall, this work aims to bridge a gap between classical theory and practical implementation by offering a comprehensive, reproducible, and rigorously validated treatment of the CSCN distribution.

Monte Carlo simulations in R statistically corroborated these derivations through Kolmogorov-Smirnov testing, convergence of estimation errors at the theoretical $1/\sqrt{N}$ rate, and comparisons between proper and improper complex Gaussian models.

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