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Mathematical Modeling For Recycling Of Sewage Water

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ABSTRACT:
Water is essential for households and industrial activities. Both produce waste water which can be treated as sewage. This sewage causes water pollution. Therefore, it is necessary to recycle it and can be treated as reclaimed water. The objective of this paper is to study the optimum recycle of sewage water. To study this recycling process of clean and sewage water, the system of non-linear differential equations is formulated. The controls are essential on untreated sewage water produced by households and industries. For validating the proposed formulation, numerical simulation is carried out.
KEYWORDS: Global Stability, Local stability, Mathematical model, Optimal control, Recycling, Sewage water, Threshold

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I. INTRODUCTION

All plants and animals must need water to survive. If there was no water, there would be no life on the earth. Apart from drinking, to live the life people have many other uses of water like cooking, washing, cleaning, keeping plants alive and also can be use in all kind of industry. But when we use water for different needs it contains germs, worms or toxic chemicals which makes it unsafe. Though we have limited source of clean water, we try to use reclaimed water. Reclaimed water is the process of converting waste water into water that can be reused by establishing water recycling plants. Reclaimed water fulfilled certain needs in household, businesses, industry. It could even be treated to match drinking water standards.

Many researchers have started research on the recycling of wastewater. In 2003, [1] has worked on dynamic modeling and simulation of water environment management with a focus on water recycling. [2] has studied water localisation and reclamation: steps towards low impact urban design and development in 2007. In 2006, [3] analysed reuse of effluent water-benefits and risks. A mathematical programming model for water usage and treatment network design was prepared by [4] in 1999. [5] deliberated agricultural reuse of wastewater: nation-wide cost-benefit analysis in 1997. Mathematical model for analysis of recirculating vertical flow constructed wetlands was proposed in 2010 by [6]. In 2004, [7] investigated activated sludge wastewater treatment plant modelling and simulation. To reduce water pollution, some researchers have studied the various mathematical models. [8] developed a mathematical model of water pollution control using the finite element method in 2006. [9] examined mathematical modeling on water pollution and self-purification of river Ganges. In 1985, mathematical models for nonpoint water pollution control was studied by [10]. Also, we can save our environment by reviving forest. Some researchers have studied this kind mathematical modeling of forest resources [11] and green belt [12].

In the next section, we will discuss about mathematical modeling of recycling of sewage water. Equilibria will be carried out in the section 3. In the section 4, optimal control for the recycling of sewage water is evaluated. Numerical simulation is validated for transmission of sewage water in the section 5.

II. MATHEMATICAL MODELING

We live in the society where water is one of the most important substance. Everyone in the society use water in each activity. For some activity, the clean water is must. But after use, it converts into waste water known as sewage water. This water is also produced by industry. Some percentage of sewage water is treatable and some is untreatable. Treated sewage water is renewed into reclaimed water. Some of the portion of reclaimed water can be used in the place of clean water. This is a cyclic process. Therefore, to examine this cycle, we have considered six discrete compartments viz., the cubic usage of clean water (W_C), the cubic usage of reclaimed water (W_R), the cubic water used in household (H), the cubic water used in industry (I), the cubic untreated

sewage water (S_U) and the cubic treated sewage water (S_T). To treat as much as sewage water, we have taken u_1 and u_2 as the controls on household and industry which produce untreated sewage water.
The notations and parametric values used for dynamic model of recycling of sewage water is give in the following table 1.

Table 1. Notations and parametric values

Notation			Parametric value
B_1	:	Recruitment rate of clean water through rain	0.8
B_2	:	Recruitment rate of recycled water through sea	0.4
β_1	:	The rate of clean water used by humans	0.5
δ_1	:	The rate of clean water used by industries	2.3
β_2	:	The rate of recycle water used by humans	0.5
δ_2	:	The rate of recycle water used by industries	0.5
η_1	:	The rate of untreated sewage water received from humans	0.2
γ_1	:	The rate of treated sewage water received from humans	0.7
η_2	:	The rate of untreated sewage water received from industries	0.7
γ_2	:	The rate of treated sewage water received from industries	0.2
ε	:	The rate of treatable sewage water for recycling	0.15
α	:	The rate of reclaimed water which can be used as clean water	0.3
μ	:	Natural rate of waste water from each compartment	0.2

Assuming some necessary assumptions and using above parameters, we have formulated the mathematical model for recycling of sewage water whose transmission diagram as in figure 1.

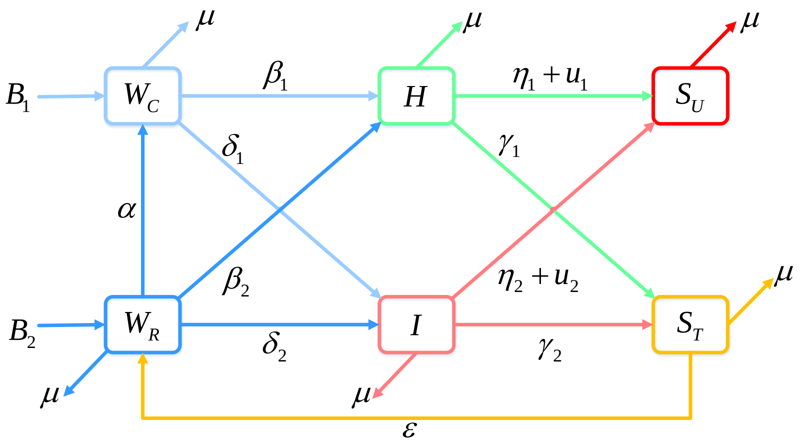

Figure 1. Transmission diagram of recycling of sewage water

The dynamics of sewage transmission in clean and reclaimed water along with household and industrial use is given as below:

$$\frac{dW_C}{dt} = B_1 - \beta_1 W_C H - \delta_1 W_C I + \alpha W_R - \mu W_C \quad (1)$$

$$\frac{dW_R}{dt} = B_2 - \beta_2 W_R H - \delta_2 W_R I - \alpha W_R + \varepsilon S_T W_R - \mu W_R \quad (2)$$

$$\frac{dH}{dt} = \beta_1 W_C H + \beta_2 W_R H - \eta_1 H S_U - \gamma_1 H S_T - u_1 H - \mu H \quad (3)$$

$$\frac{dI}{dt} = \delta_1 W_C I + \delta_2 W_R I - \eta_2 I S_U - \gamma_2 I S_T - u_2 I - \mu I \quad (4)$$

$$\frac{dS_U}{dt} = \eta_1 H S_U + \eta_2 I S_U + u_1 H + u_2 I - \mu S_U \quad (5)$$

$$\frac{dS_T}{dt} = \gamma_1 H S_T + \gamma_2 I S_T - \varepsilon S_T W_R - \mu S_T \quad (6)$$

with $W_C + W_R + H + I + S_U + S_T = N$ and $W_C > 0; W_R, H, I, S_U, S_T \geq 0$.
Adding above all differential equations of the model, we get

$$\begin{aligned}\frac{d}{dt}\left(W_C + W_R + H + I + S_U + S_R\right) &= B_1 - \beta_1 W_C H - \delta_1 W_C I + \alpha W_R - \mu W_C + B_2 - \beta_2 W_R H - \delta_2 W_R I - \alpha W_R + \varepsilon S_T W_R - \mu W_R \\ &\quad + \beta_1 W_C H + \beta_2 W_R H - \eta_1 H S_U - \gamma_1 H S_T - u_1 H - \mu H + \delta_1 W_C I + \delta_2 W_R I - \eta_2 I S_U \\ &\quad - \gamma_2 I S_T - u_2 I - \mu I + \eta_1 H S_U + \eta_2 I S_U + u_1 H + u_2 I - \mu S_U + \gamma_1 H S_T + \gamma_2 I S_T - \varepsilon S_T W_R \\ &\quad - \mu S_T \\ &= B_1 + B_2 - \mu\left(W_C + W_R + H + I + S_U + S_R\right)\end{aligned}$$

which gives us, $\lim_{t\to\infty}\sup\left(W_C + W_R + H + I + S_U + S_R\right) \leq \frac{B_1 + B_2}{\mu}$

Hence, the feasible region of the model is

$$\Lambda = \left\{\left(W_C + W_R + H + I + S_U + S_R\right) / W_C + W_R + H + I + S_U + S_R \leq \frac{B_1 + B_2}{\mu}\right\}$$

Now, from equation (5), we get

$$\left(\eta_1 H + \eta_2 I - \mu\right) S_U = 0$$

$$\Rightarrow \eta_1 H + \eta_2 I - \mu = 0 \tag{7}$$

From equation (6), we get

$$\left(\gamma_1 H + \gamma_2 I - \varepsilon W_R - \mu\right) S_T = 0$$

$$\Rightarrow \gamma_1 H + \gamma_2 I - \varepsilon \frac{B_2}{\mu} - \mu = 0 \tag{8}$$

From equation (7) and (8), we get

$$H = \frac{-\mu^2\gamma_2 + \mu^2\eta_2 + \varepsilon B_2 \eta_2}{\mu\left(-\gamma_2\eta_1 + \gamma_1\eta_2\right)} \text{ and } I = \frac{-\mu^2\gamma_1 + \mu^2\eta_1 + \varepsilon B_2\eta_1}{\mu\left(\gamma_2\eta_1 - \gamma_1\eta_2\right)}$$

Putting the values of H and I in equation (4), we find

$$S_U = \frac{-\mu^2 + B_1\delta_1 + B_2\delta_2}{\mu\eta_2}$$

Thus, equilibrium point $E^*\left(W_C^*, W_R^*, H^*, I^*, S_U^*, S_T^*\right)$ is

$$E^*\left(\frac{B_1}{\mu}, \frac{B_2}{\mu}, \frac{-\mu^2\gamma_2 + \mu^2\eta_2 + \varepsilon B_2\eta_2}{\mu\left(-\gamma_2\eta_1 + \gamma_1\eta_2\right)}, \frac{-\mu^2\gamma_1 + \mu^2\eta_1 + \varepsilon B_2\eta_1}{\mu\left(\gamma_2\eta_1 - \gamma_1\eta_2\right)}, \frac{-\mu^2 + B_1\delta_1 + B_2\delta_2}{\mu\eta_2}, 0\right)$$

Now, using next generation matrix method, we compute basic reproduction number R_0.
Let, $X' = \left(W_C, W_R, H, I, S_U, S_T\right)'$, where dash denotes the derivatives. So,

$$X' = \frac{dF}{dt} = F\left(X\right) - V\left(X\right)$$

where $F\left(X\right)$ is the rate of appearance of new recruitment in the component and $V\left(X\right)$ is the rate of transfer of sewage water. They are given by

$$F = \begin{bmatrix} \beta_1 W_C H + \beta_2 W_R H \\ \delta_1 W_C I + \delta_2 W_R I \\ \eta_1 H S_U + \eta_2 I S_U \\ \gamma_1 H S_T + \gamma_2 I S_T \\ \varepsilon S_T W_R \\ 0 \end{bmatrix} \text{ and } V = \begin{bmatrix} \eta_1 H S_U + \gamma_1 H S_T + \mu H \\ \eta_2 I S_U + \gamma_2 I S_T + \mu I \\ \mu S_U \\ \varepsilon S_T W_R + \mu S_T \\ -B_1 + \beta_1 W_C H + \delta_1 W_C I - \alpha W_R + \mu W_C \\ -B_2 + \beta_2 W_R H + \delta_2 W_R I + \alpha W_R + \mu W_R \end{bmatrix}$$

Now, $DF\left(E^*\right) = \begin{bmatrix} f & 0 \\ 0 & 0 \end{bmatrix}$ and $DV\left(E^*\right) = \begin{bmatrix} v & 0 \\ J_1 & J_2 \end{bmatrix}$

where f and v are 6×6 matrices defined as

$f = \left[\dfrac{\partial F_i\left(E^*\right)}{\partial X_j} \right]$ and $v = \left[\dfrac{\partial V_i\left(E^*\right)}{\partial X_j} \right]$

Which gives us,

$$f = \begin{bmatrix} \beta_1 W_C^* + \beta_2 W_R^* & 0 & 0 & 0 & \beta_2 H^* & \beta_1 H^* \\ 0 & \delta_1 W_C^* + \delta_2 W_R^* & 0 & 0 & \delta_2 I^* & \delta_1 I^* \\ \eta_1 S_U^* & \eta_2 S_U^* & \eta_1 H^* + \eta_2 I^* & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma_1 H^* + \gamma_2 I^* & 0 & 0 \\ 0 & 0 & 0 & \varepsilon W_R^* & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$v = \begin{bmatrix} \eta_1 S_U^* + \mu & 0 & \eta_1 H^* & \gamma_1 H^* & 0 & 0 \\ 0 & \eta_2 S_U^* + \mu & \eta_2 I^* & \gamma_2 I^* & 0 & 0 \\ 0 & 0 & \mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ \beta_2 W_R^* & \delta_2 W_R^* & 0 & 0 & \beta_2 H^* + \delta_2 I^* + \alpha + \mu & 0 \\ \beta_1 W_C^* & \beta_2 W_C^* & 0 & 0 & -\alpha & \beta_1 H^* + \delta_1 I^* + \mu \end{bmatrix}$$

where $W_C^* = \dfrac{B_1}{\mu}$, $W_R^* = \dfrac{B_2}{\mu}$, $H^* = \dfrac{-\mu^2\gamma_2 + \mu^2\eta_2 + \varepsilon B_2\eta_2}{\mu\left(-\gamma_2\eta_1 + \gamma_1\eta_2\right)}$, $I^* = \dfrac{-\mu^2\gamma_1 + \mu^2\eta_1 + \varepsilon B_2\eta_1}{\mu\left(\gamma_2\eta_1 - \gamma_1\eta_2\right)}$,

$$S_U^* = \frac{-\mu^2 + B_1\delta_1 + B_2\delta_2}{\mu\eta_2} \text{ and } S_T^* = 0 \tag{9}$$

Here, v is non-singular matrix. So, the basic reproduction number R_0 is as follows:

$R_0 =$ spectral radius of matrix fv^{-1}

$$R_0 = \frac{\delta_1 W_C^* + \delta_2 W_R^*}{\left(\eta_2 S_U^* + \mu\right)} - \frac{\delta_2^2 I^* W_R^*}{\left(\eta_2 S_U^* + \mu\right)\left(\beta_2 H^* + \delta_2 I^* + \alpha + \mu\right)} - \frac{\delta_1^2 I W_C^*\left(\beta_2 H^* + \delta_2 I^* + \alpha + \mu\right) + \alpha\delta_1\delta_2 I^* W_R^*}{\left(\eta_2 S_U^* + \mu\right)\left(\beta_2 H^* + \delta_2 I^* + \alpha + \mu\right)\left(\beta_1 H^* + \delta_1 I^* + \mu\right)} \tag{10}$$

The stability analysis of the transmission model for recycling of sewage water will be discussed in the sext section.

III. EQUILIBRIUM

The equilibrium of local stability and global stability of the transmission of sewage water model will be discussed in this section.

3.1. Local Stability

Here, we determine the local stability of the model for recycling of sewage water.

Theorem 1: The unique positive equilibrium point $E^*\left(W_C^*, W_R^*, H^*, I^*, S_U^*, S_T^*\right)$ of the transmission of sewage water model is locally asymptotically stable with the conditions $\delta_1\eta_1 > \beta_1\eta_2$ and $\beta_2\eta_2 > \delta_2\eta_1$.

Proof: Here, we will examine the local stability of the model for recycling of sewage water for equilibrium point $E^*\left(W_C^*, W_R^*, H^*, I^*, S_U^*, S_T^*\right)$ by using the Jacobian matrix J where $W_C^*, W_R^*, H^*, I^*, S_U$ and S_T^* are defined as in equation (9).

The Jacobian matrix of the model is as follows:

$$J=\begin{bmatrix} \left(-\beta_1 H^*-\delta_1 I^*-\mu\right) & \alpha & -\beta_1 W_C^* & -\delta_1 W_C^* & 0 & 0 \\ 0 & \begin{pmatrix}-\beta_2 H^*-\delta_2 I^* \\ -\alpha-\mu\end{pmatrix} & -\beta_2 W_R^* & -\delta_2 W_R^* & 0 & \varepsilon W_R^* \\ \beta_1 H^* & \beta_2 H^* & \begin{pmatrix}\beta_1 W_C^*+\beta_2 W_R^* \\ -\eta_1 S_U^*-u_1-\mu\end{pmatrix} & 0 & -\eta_1 H^* & -\gamma_1 H^* \\ \delta_1 I^* & \delta_2 I^* & 0 & \begin{pmatrix}\delta_1 W_C^*+\delta_2 W_R^* \\ -\eta_2 S_U^*-u_2-\mu\end{pmatrix} & -\eta_2 I^* & -\gamma_2 I^* \\ 0 & 0 & \eta_1 S_U^*+u_1 & \eta_2 S_U^*+u_2 & \left(\eta_1 H^*+\eta_2 I^*-\mu\right) & 0 \\ 0 & 0 & 0 & 0 & 0 & \begin{pmatrix}\gamma_1 H^*+\gamma_2 I^* \\ -\varepsilon W_R^*-\mu\end{pmatrix} \end{bmatrix}$$

Now, taking $a_{11}=\beta_1 H^*+\delta_1 I^*+\mu$, $a_{22}=\beta_2 H^*+\delta_2 I^*+\alpha+\mu$, $a_{33}=-\beta_1 W_C^*-\beta_2 W_R^*+\eta_1 S_U^*+u_1+\mu$, $a_{44}=-\delta_1 W_C^*-\delta_2 W_R^*+\eta_2 S_U^*+u_2+\mu$, $a_{55}=-\eta_1 H^*-\eta_2 I^*+\mu$, $a_{66}=-\gamma_1 H^*-\gamma_2 I^*+\varepsilon W_R^*+\mu$

Then the Jacobian matrix is as follows:

$$J=\begin{bmatrix} -a_{11} & \alpha & -\beta_1 W_C^* & -\delta_1 W_C^* & 0 & 0 \\ 0 & -a_{22} & -\beta_2 W_R^* & -\delta_2 W_C^* & 0 & \varepsilon W_R^* \\ \beta_1 H^* & \beta_2 H^* & -a_{33} & 0 & -\eta_1 H^* & -\gamma_1 H^* \\ \delta_1 I^* & \delta_2 I^* & 0 & -a_{44} & -\eta_2 I^* & -\gamma_2 I^* \\ 0 & 0 & \eta_1 S_U^*+u_1 & \eta_2 S_U^*+u_2 & -a_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & -a_{66} \end{bmatrix}$$

Clearly, one of the eigen value of above Jacobian matrix is $\gamma_1 H^*+\gamma_2 I^*+\varepsilon W_R^*-\mu$. So, sub matrix of Jacobian matrix J is J_1 as follows:

$$J_1=\begin{bmatrix} -a_{11} & \alpha & -\beta_1 W_C^* & -\delta_1 W_C^* & 0 \\ 0 & -a_{22} & -\beta_2 W_R^* & -\delta_2 W_C^* & 0 \\ \beta_1 H^* & \beta_2 H^* & -a_{33} & 0 & -\eta_1 H^* \\ \delta_1 I^* & \delta_2 I^* & 0 & -a_{44} & -\eta_2 I^* \\ 0 & 0 & \eta_1 S_U^*+u_1 & \eta_2 S_U^*+u_2 & -a_{55} \end{bmatrix}$$

Now, the associated characteristics equation of Jacobian matrix J_1 is

$$\lambda^5+A_1\lambda^4+A_2\lambda^3+A_3\lambda^2+A_4\lambda+A_5=0$$

where

$$A_{11}=a_{55}+a_{44}+a_{33}+a_{22}+a_{11}$$

$$\begin{aligned} A_{22}=A_{22}&=\eta_1^2 H^* S_U^*+\eta_1 H^* u_1+\eta_2^2 I^* S_U^*+\eta_2 I^* u_2+a_{55}a_{44}+a_{55}a_{33}+a_{55}a_{22}+a_{55}a_{11}+\delta_1^2 W_C^* I^*+\delta_2^2 W_R^* I^*+a_{44}a_{33} \\ &\quad +a_{44}a_{22}+a_{44}a_{11}+\beta_1^2 W_C^* H^*+\beta_2^2 W_R^* H^*+a_{33}a_{22}+a_{33}a_{11}+a_{22}a_{11} \\ &=a_{55}\left(a_{44}+a_{33}+a_{22}+a_{11}\right)+a_{44}\left(a_{33}+a_{22}+a_{11}\right)+a_{33}\left(a_{22}+a_{11}\right)+a_{22}\left(a_{11}\right)+H^*\left(\beta_1^2 W_C^*+\beta_2^2 W_R^*\right)+I^*\left(\delta_1^2 W_C^*\right. \\ &\quad \left.+\delta_2^2 W_R^*\right)+\eta_1 H^*\left(\eta_1 S_U^*+u_1\right)+\eta_2 I^*\left(\eta_2 S_U^*+u_2\right) \end{aligned}$$

$$
\begin{aligned}
A_{33} &= a_{55}a_{44}a_{33} + a_{55}a_{44}a_{22} + a_{55}a_{44}a_{11} + a_{55}a_{33}a_{22} + a_{55}a_{33}a_{11} + a_{55}a_{22}a_{11} + a_{44}a_{33}a_{22} + a_{44}a_{33}a_{11} + a_{44}a_{22}a_{11} + a_{33}a_{22}a_{11} + a_5\beta_1^2 W_C^* H^* \\
&\quad + a_5\beta_2^2 W_R^* H^* + a_5\delta_1^2 W_C^* I^* + a_5\delta_2^2 W_R^* I^* + a_4\beta_1^2 W_C^* H^* + a_4\beta_2^2 W_R^* H^* + a_4\eta_1^2 H^* S_U^* + a_4\eta_1 H^* u_1 + a_3\delta_1^2 W_C^* I^* + a_3\delta_2^2 W_R^* I \\
&\quad + a_3\eta_2^2 I^* S_U^* + a_3\eta_2 I^* u_2 + a_2\eta_1^2 H^* S_U^* + a_2\eta_1 H^* u_1 + a_2\eta_2^2 I^* S_U^* + a_2\eta_2 I^* u_2 + a_1\eta_1^2 H^* S_U^* + a_1\eta_1 H^* u_1 + a_1\eta_2^2 I^* S_U^* + a_1\eta_2 I^* u_2 \\
&\quad + a_2\beta_1^2 W_C^* H^* + a_1\beta_2^2 W_R^* H^* + a_2\delta_1^2 W_C^* I^* + a_1\delta_2^2 W_R^* I^* + \beta_1\beta_2\alpha W_R^* H^* + \delta_1\delta_2\alpha W_R^* I^* \\
&= a_{55}a_{44}(a_{33} + a_{22} + a_{11}) + a_{55}a_{33}(a_{22} + a_{11}) + a_{55}a_{22}(a_{11}) + a_{44}a_{33}(a_{22} + a_{11}) + a_{44}a_{22}(a_{11}) + a_{33}a_{22}(a_{11}) + \alpha W_R^*(\beta_1\beta_2 H^* + \delta_1\delta_2 I^*) \\
&\quad + a_{55}(H^*(\beta_1^2 W_C^* + \beta_2^2 W_R^*) + I^*(\delta_1^2 W_C^* + \delta_2^2 W_R^*)) + a_{44}H^*(\beta_1^2 W_C^* + \beta_2^2 W_R^* + \eta_1(\eta_1 S_U^* + u_1)) + a_{33}I^*(\delta_1^2 W_C^* + \delta_2^2 W_R^* \\
&\quad + \eta_2(\eta_2 S_U^* + u_2)) + a_{22}(H^*(\beta_1^2 W_C^* + \eta_1(\eta_1 S_U^* + u_1)) + I^*(\delta_1^2 W_C^* + \eta_2(\eta_2 S_U^* + u_2))) + a_{11}(H^*(\beta_2^2 W_R^* + \eta_1(\eta_1 S_U^* + u_1) \\
&\quad + I(\delta_2^2 W_R^* + \eta_2(\eta_2 S_U^* + u_2))))
\end{aligned}
$$

$$
\begin{aligned}
A_{44} &= a_{55}a_{44}a_{33}a_{22} + a_{55}a_{44}a_{33}a_{11} + a_{55}a_{44}a_{22}a_{11} + a_{55}a_{33}a_{22}a_{11} + a_{44}a_{33}a_{22}a_{11} + a_{55}a_{44}\beta_1^2 W_C^* H^* + a_{55}a_{44}\beta_2^2 W_R^* H^* + a_{55}a_{33}\delta_1^2 W_C^* I^* \\
&\quad + a_{55}a_{33}\delta_2^2 W_R^* I^* + a_{55}a_{22}\beta_1^2 W_C^* H^* + a_{55}a_{11}\beta_2^2 W_R^* H^* + a_{55}a_{22}\delta_1^2 W_C^* I^* + a_{55}a_{11}\delta_2^2 W_R^* I^* + a_{44}a_{22}\beta_1^2 W_C^* H^* + a_{44}a_{11}\beta_2^2 W_R^* H^* \\
&\quad + a_{33}a_{22}\delta_1^2 W_C^* I^* + a_{33}a_{11}\delta_2^2 W_R^* I^* + a_{44}a_{22}\eta_1^2 H^* S_U^* + a_{44}a_{22}\eta_1 H^* u_1 + a_{44}a_{11}\eta_1^2 H^* S_U^* + a_{44}a_{11}\eta_1 H^* u_1 + a_{22}a_{11}\eta_1^2 H^* S_U^* \\
&\quad + a_{22}a_{11}\eta_1 H^* u_1 + a_{33}a_{22}\eta_2^2 I^* S_U^* + a_{33}a_{22}\eta_2 I^* u_2 + a_{33}a_{11}\eta_2^2 I^* S_U^* + a_{33}a_{11}\eta_2 I^* u_2 + a_{22}a_{11}\eta_2^2 I^* S_U^* + a_{22}a_{11}\eta_2 I^* u_2 + \delta_1^2\eta_1^2 W_C^* H^* I^* S_U^* \\
&\quad + \delta_1^2\eta_1 W_C^* H^* I^* u_1 + \delta_2^2\eta_1^2 W_R^* H^* I^* S_U^* + \delta_2^2\eta_1 W_R^* H^* I^* u_1 + \beta_1^2\eta_2^2 W_C^* H^* I^* S_U^* + \beta_1^2\eta_2 W_C^* H^* I^* u_2 + \beta_2^2\eta_2^2 W_R^* H^* I^* S_U^* \\
&\quad + \beta_2^2\eta_2 W_R^* H^* I^* u_2 + a_{55}\beta_1\beta_2\alpha W_R^* H^* + a_{44}\beta_1\beta_2\alpha W_R^* H^* + a_{55}\delta_1\delta_2\alpha W_R^* I^* + a_{33}\delta_1\delta_2\alpha W_R^* I^* + \beta_1^2\delta_2^2 W_C^* W_R^* H^* I^* \\
&\quad - \beta_1\beta_2\delta_1\delta_2 W_C^* W_R^* H^* I^* + \beta_2^2\delta_1^2 W_C^* W_R^* H^* I^* - \beta_1\beta_2\delta_1\delta_2 W_C^* W_R^* H^* I^* - \beta_1\delta_1\eta_1\eta_2 W_C^* H^* I^* S_U^* - \beta_1\delta_1\eta_2 W_C^* H^* I^* u_1 \\
&\quad - \beta_1\delta_1\eta_1\eta_2 W_C^* H^* I^* S_U^* - \beta_1\delta_1\eta_1 W_C^* H^* I^* u_2 - \beta_2\delta_2\eta_1\eta_2 W_R^* H^* I^* S_U^* - \beta_2\delta_2\eta_2 W_R^* H^* I^* u_1 - \beta_2\delta_2\eta_1\eta_2 W_R^* H^* I^* S_U^* - \beta_2\delta_2\eta_1 W_R^* H^* I^* u_2 \\
&= a_{55}a_{44}a_{33}(a_{22} + a_{11}) + a_{55}a_{44}a_{22}(a_{11}) + a_{55}a_{33}a_{22}(a_{11}) + a_{44}a_{33}a_{22}(a_{11}) + a_{55}a_{44}H^*(\beta_1^2 W_C^* + \beta_2^2 W_R^*) + a_{55}a_{33}I^*(\delta_1^2 W_C^* + \delta_2^2 W_R^*) \\
&\quad + (a_{55} + a_{44})H^*(\beta_1\beta_2\alpha W_R^* H^* + a_{22}\beta_1^2 W_C^* + a_{11}\beta_2^2 W_R^*) + (a_{55} + a_{33})I^*(\delta_1\delta_2\alpha W_R^* I^* + a_{22}\delta_1^2 W_C^* + a_{11}\delta_2^2 W_R^*) + W_C^* W_R^* H^* I^*(\beta_1\delta_2 \\
&\quad - \beta_2\delta_1)^2 + (a_{44}a_{22} + a_{44}a_{11} + a_{22}a_{11})\eta_1 H^*(\eta_1 S_U^* + u_1) + (a_{33}a_{22} + a_{33}a_{11} + a_{22}a_{11})\eta_2 I^*(\eta_2 S_U^* + u_2) + H^* I^*(\eta_1 S_U^* + u_1)(\delta_1 W_C^*(\delta_1\eta_1 \\
&\quad - \beta_1\eta_2) + \delta_2 W_R^*(\beta_2\eta_2 - \delta_2\eta_1)) + H^* I^*(\eta_2 S_U^* + u_2)(\beta_1 W_C^*(\delta_1\eta_1 - \beta_1\eta_2) + \beta_2 W_R^*(\beta_2\eta_2 - \delta_2\eta_1))
\end{aligned}
$$

$$
\begin{aligned}
A_{55} &= a_{55}a_{44}a_{33}a_{22}a_{11} + a_{55}a_{44}a_{22}\beta_1^2 W_C^* H^* + a_{55}a_{44}a_{11}\beta_2^2 W_R^* H^* + a_{55}a_{33}a_{22}\delta_1^2 W_C^* I^* + a_{55}a_{33}a_{11}\delta_2^2 W_R^* I^* + a_{55}a_{44}\beta_1\beta_2\alpha W_R^* H^* \\
&\quad + a_{55}a_{33}\delta_1\delta_2\alpha W_R^* I^* + a_{55}\beta_1^2\delta_2^2 W_C^* W_R^* H^* I^* - a_{55}\beta_1\beta_2\delta_1\delta_2 W_C^* W_R^* H^* I^* + a_{55}\beta_2^2\delta_1^2 W_C^* W_R^* H^* I^* - a_{55}\beta_1\beta_2\delta_1\delta_2 W_C^* W_R^* H^* I^* \\
&\quad + a_{44}a_{22}a_{11}\eta_1^2 H^* S_U^* + a_{44}a_{22}a_{11}\eta_1 H^* u_1 + a_{33}a_{22}a_{11}\eta_2^2 I^* S_U^* + a_{33}a_{22}a_{11}\eta_2 I^* u_1 + a_{22}\delta_1^2\eta_1^2 W_C^* H^* I^* S_U^* + a_{22}\delta_1^2\eta_1 W_C^* H^* I^* u_1 \\
&\quad + a_{11}\delta_2^2\eta_1^2 W_R^* H^* I^* S_U^* + a_{11}\delta_2^2\eta_1 W_R^* H^* I^* u_1 + a_{22}\beta_1^2\eta_2^2 W_C^* H^* I^* S_U^* + a_{22}\beta_1^2\eta_2 W_C^* H^* I^* u_2 + a_{11}\beta_2^2\eta_2^2 W_R^* H^* I^* S_U^* \\
&\quad + a_{11}\beta_2^2\eta_2 W_R^* H^* I^* u_2 - a_{22}\beta_1\delta_1\eta_1\eta_2 W_C^* H^* I^* S_U^* - a_{22}\beta_1\delta_1\eta_2 W_C^* H^* I^* u_1 - a_{22}\beta_1\delta_1\eta_1\eta_2 W_C^* H^* I^* S_U^* - a_{22}\beta_1\delta_1\eta_1 W_C^* H^* I^* u_2 \\
&\quad - a_{11}\beta_2\delta_2\eta_1\eta_2 W_R^* H^* I^* S_U^* - a_{11}\beta_2\delta_2\eta_2 W_R^* H^* I^* u_1 - a_{11}\beta_2\delta_2\eta_1\eta_2 W_R^* H^* I^* S_U^* - a_{11}\beta_2\delta_2\eta_1 W_R^* H^* I^* u_2 - \beta_1\delta_2\eta_1\eta_2\alpha W_R^* H^* I^* S_U^* \\
&\quad - \beta_1\delta_2\eta_2\alpha W_R^* H^* I^* u_1 - \beta_2\delta_1\eta_1\eta_2\alpha W_R^* H^* I^* S_U^* - \beta_2\delta_1\eta_1\alpha W_R^* H^* I^* u_2 + \delta_1\delta_2\eta_1^2\alpha W_R^* H^* I^* S_U^* + \delta_1\delta_2\eta_1\alpha W_R^* H^* I^* u_1 \\
&\quad + \beta_1\beta_2\eta_2^2\alpha W_R^* H^* I^* S_U^* + \beta_1\beta_2\eta_2\alpha W_R^* H^* I^* u_2 \\
&= a_{55}a_{44}a_{33}a_{22}(a_{11}) + a_{44}a_{22}a_{11}\eta_1 H^*(\eta_1 S_U^* + u_1) + a_{33}a_{22}a_{11}\eta_2 I^*(\eta_2 S_U^* + u_2) + a_{55}a_{44}H^*(a_{22}\beta_1^2 W_C^* + a_{11}\beta_2^2 W_R^*) + a_{55}a_{33}I^*(a_{22}\delta_1^2 W_C^* \\
&\quad + a_{11}\delta_2^2 W_R^*) + a_{55}\alpha W_R^*(a_{44}\beta_1\beta_2 H^* + a_{33}\delta_1\delta_2 I^*) + a_{55}W_C^* W_R^* H^* I^*(\beta_1\delta_2 - \beta_2\delta_1)^2 + H^* I^*(\eta_1 S_U^* + u_1)((a_2\delta_1 W_C^* + \delta_2\alpha W_R^*)(\delta_1\eta_1 \\
&\quad - \beta_1\eta_2) + a_1\delta_2 W_R^*(\beta_2\eta_2 - \delta_2\eta_1)) + H^* I^*(\eta_2 S_U^* + u_2)((a_2\beta_1 W_C^* + \beta_2\alpha W_R^*)(\delta_1\eta_1 - \beta_1\eta_2) + a_1\beta_2 W_R^*(\beta_2\eta_2 - \delta_2\eta_1))
\end{aligned}
$$

Hence, $\delta_1\eta_1 - \beta_1\eta_2 > 0$ if $\delta_1\eta_1 > \beta_1\eta_2$ and $\beta_2\eta_2 - \delta_2\eta_1 > 0$ if $\beta_2\eta_2 > \delta_2\eta_1$ which proves the stability of the system.

3.2. Global Stability

Here, we govern the global stability of the model for recycling of sewage water.

Theorem 3: The unique positive equilibrium point $E^*(W_C^*, W_R^*, H^*, I^*, S_U^*, S_T^*)$ of the transmission of sewage water model is globally asymptotically stable without any conditions.

Proof: Consider a Lyapunov function $L(t)$.

$$L(t) = \frac{1}{2}\left[(W_C - W_C^*) + (W_R - W_R^*) + (H - H^*) + (I + I^*) + (S_U - S_U^*) + (S_T - S_T^*)\right]^2$$

$$
\begin{aligned}
L'(t) &= \left[\left(W_C - W_C^*\right) + \left(W_R - W_R^*\right) + \left(H - H^*\right) + \left(I + I^*\right) + \left(S_U - S_U^*\right) + \left(S_T - S_T^*\right)\right]\left[W_C' + W_R' + H' + I' + S_U' + S_T'\right] \\
&= \left[\left(W_C - W_C^*\right) + \left(W_R - W_R^*\right) + \left(H - H^*\right) + \left(I + I^*\right) + \left(S_U - S_U^*\right) + \left(S_T - S_T^*\right)\right] \\
&\qquad \left[\mu W_C^* + \mu W_R^* + \mu H^* + \mu I^* + \mu S_U^* + \mu S_T^* + \mu W_C + \mu W_R + \mu H + \mu I + \mu S_U + \mu S_T\right] \\
&= -\mu\left[\left(W_C - W_C^*\right) + \left(W_R - W_R^*\right) + \left(H - H^*\right) + \left(I + I^*\right) + \left(S_U - S_U^*\right) + \left(S_T - S_T^*\right)\right]^2 \le 0
\end{aligned}
$$

We have taken $B_1 + B_2 = \mu W_C^* + \mu W_R^* + \mu H^* + \mu I^* + \mu S_U^* + \mu S_T^*$

Therefore, E^* is globally stable.

IV. OPTIMAL CONTROL

The objective of the model is to minimize the cubic untreated sewage water so that we could use more cubic reclaimed water. The control functions are united to achieve the objective. The objective function for the mathematical model of recycling of sewage water along with the optimal control is given by

$$
J(u_i, \Omega) = \int_0^T \left(A_1 W_C^2 + A_2 W_R^2 + A_3 H^2 + A_4 I^2 + A_5 S_U^2 + A_6 S_T^2 + w_1 u_1^2 + w_2 u_2^2\right) dt \qquad (11)
$$

where, Ω denotes set of all compartmental variables, A_1, A_2, A_3, A_4, A_5 denote non-negative weight constants for W_C, W_R, H, I, S_U, S_T compartments respectively and w_1, w_2 are weight constants for control variables u_1, u_2 respectively.

As, the weight parameters w_1 and w_2 are constants of the control rate for untreated sewage water received from household (u_1) and the control rate of untreated sewage water received from industry (u_2) respectively, from which the optimal control condition is normalized. u_1 and u_2 both the control rates are for minimizing the cubic density of untreated sewage water. Now, we will calculate the values of control variables u_1 and u_2 from $t = 0$ to $t = T$ such that

$J(u_1(t), u_2(t)) = \min\{J(u_i^*, \Omega) / (u_1, u_2) \in \phi\}$

where ϕ is a smooth function on the interval $[0,1]$. The optimal controls denoted by $u_i^*, i = 1,2,3$ are found by accumulating all the integrands of equation (11) using the lower bounds and upper bounds respectively with the results of Fleming and Rishel (2012).

Now, using the pontrygin's principle from Boltyanki *et al.* (1986), to minimize the cost function in (11) by constructing Lagrangian function consisting of state equations and adjoint variables $A_i = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6)$ as

$$
\begin{aligned}
L(\Omega, A_i) = {} & A_1 W_C^2 + A_2 W_R^2 + A_3 H^2 + A_4 I^2 + A_5 S_U^2 + A_6 S_T^2 + w_1 u_1^2 + w_2 u_2^2 \\
& + \lambda_1\left(B_1 - \beta_1 W_C H - \delta_1 W_C I + \alpha W_R - \mu W_C\right) \\
& + \lambda_2\left(B_2 - \beta_2 W_R H - \delta_2 W_R I - \alpha W_R + \varepsilon S_T W_R - \mu W_R\right) \\
& + \lambda_3\left(\beta_1 W_C H + \beta_2 W_R H - \eta_1 H S_U - \gamma_1 H S_T - u_1 H - \mu H\right) \\
& + \lambda_4\left(\delta_1 W_C I + \delta_2 W_R I - \eta_2 I S_U - \gamma_2 I S_T - u_2 I - \mu I\right) \\
& + \lambda_5\left(\eta_1 H S_U + \eta_2 I S_U + u_1 H + u_2 I - \mu S_U\right) \\
& + \lambda_6\left(\gamma_1 H S_T + \gamma_2 I S_T - \varepsilon S_T W_R - \mu S_T\right)
\end{aligned} \qquad (12)
$$

The partial derivative of the Lagrangian function with respect to each variable of the compartment gives the adjoint equation variables $A_i = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6)$ corresponding to the system which is as follows:

$$
\dot{\lambda}_1 = -\frac{\partial L}{\partial W_C} = -2A_1 W_C + (\lambda_1 - \lambda_3)\beta_1 H + (\lambda_1 - \lambda_4)\delta_1 I + \lambda_1 \mu \qquad (13)
$$

$$
\dot{\lambda}_2 = -\frac{\partial L}{\partial W_R} = -2A_2 W_R + (\lambda_2 - \lambda_1)\alpha + (\lambda_2 - \lambda_3)\beta_2 H + (\lambda_2 - \lambda_4)\delta_2 I + (\lambda_6 - \lambda_2)\varepsilon S_T + \lambda_2 \mu \qquad (14)
$$

$$
\dot{\lambda}_3 = -\frac{\partial L}{\partial H} = -2A_3 H + (\lambda_1 - \lambda_3)\beta_1 W_C + (\lambda_2 - \lambda_3)\beta_2 W_R + (\lambda_3 - \lambda_5)\eta_1 S_U + (\lambda_3 - \lambda_6)\gamma_1 S_T + (\lambda_3 - \lambda_5)u_1 + \lambda_3 \mu \qquad (15)
$$

$$
\dot{\lambda}_4 = -\frac{\partial L}{\partial I} = -2A_4 I + (\lambda_1 - \lambda_4)\delta_1 W_C + (\lambda_2 - \lambda_4)\delta_2 W_R + (\lambda_4 - \lambda_5)\eta_2 S_U + (\lambda_4 - \lambda_6)\gamma_2 S_T + (\lambda_4 - \lambda_5)u_2 + \lambda_4 \mu \qquad (16)
$$

$$
\dot{\lambda}_5 = -\frac{\partial L}{\partial S_U} = -2A_5 S_U + (\lambda_3 - \lambda_5)\eta_1 H + (\lambda_4 - \lambda_5)\eta_2 I + \lambda_5 \mu \qquad (17)
$$

$$\dot{\lambda}_6 = -\frac{\partial L}{\partial S_T} = -2A_6 S_T + (\lambda_6 - \lambda_2)\varepsilon W_R + (\lambda_3 - \lambda_6)\gamma_1 H + (\lambda_4 - \lambda_6)\gamma_2 I + \lambda_6 \mu \qquad (18)$$

The necessary condition for Lagrangian function L to be optimal for controls are

$$\dot{u}_1 = -\frac{\partial L}{\partial u_1} = -2W_1 u_1 + (\lambda_3 - \lambda_5) H = 0 \qquad (19)$$

$$\dot{u}_2 = -\frac{\partial L}{\partial u_2} = -2W_2 u_2 + (\lambda_4 - \lambda_5) I = 0 \qquad (20)$$

To find the values of u_1 and u_2, we solve equations (19) and (20) then we get

$$u_1 = \frac{(\lambda_3 - \lambda_5) H}{2W_1} \text{ and } u_2 = \frac{(\lambda_4 - \lambda_5) I}{2W_2} \qquad (21)$$

Thus, the required optimal control condition is computed as

$$u_1{*} = \max\left(a_1, \min\left(b_1, \frac{(\lambda_3 - \lambda_5) H}{2W_1}\right)\right) \text{and } u_2{*} = \max\left(a_2, \min\left(b_2, \frac{(\lambda_4 - \lambda_5) I}{2W_2}\right)\right) \qquad (22)$$

In next section, to find the analytical results, the optimal control is calculated numerically.

V. NUMERICAL SIMULATION

In this section, we will discuss some numerical simulation of the model.

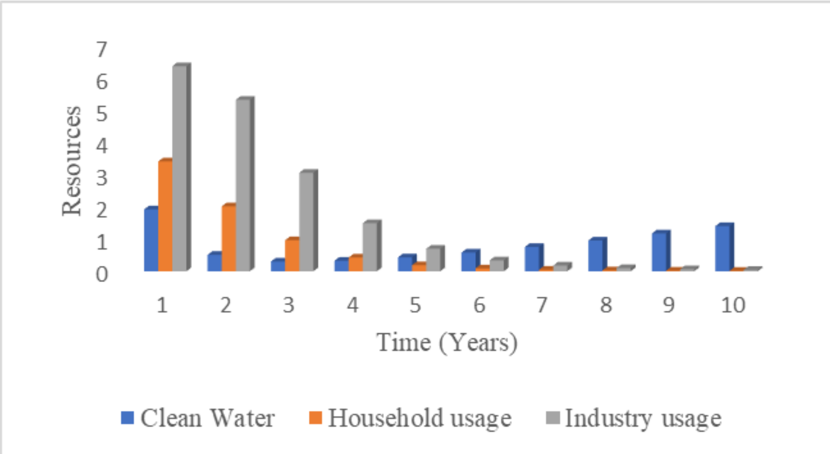

Figure 2. Cubic usage of clean water in household and industry

Above figure shows the cubic usage of clean water. Clearly, the usage of clean water is decreasing for household and industry with the time. Also, the cubic density of clean water decreases for first three years and after that it increases with the time as we use reclaimed water instead of clean water.

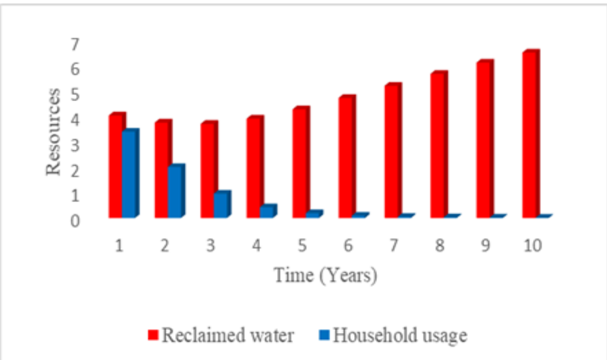

Figure 3. Cubic usage of reclaimed water in household

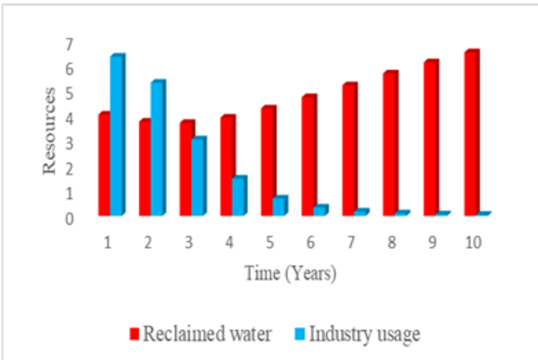

Figure 4. Cubic usage of reclaimed water in household

Figure 3 indicates the effect for usage of reclaimed water in household. Here, as the household usage is increasing, the cubic volume of reclaimed water is increasing which says that we should treat more reclaimed water. Similar effect is analysed in figure 4.

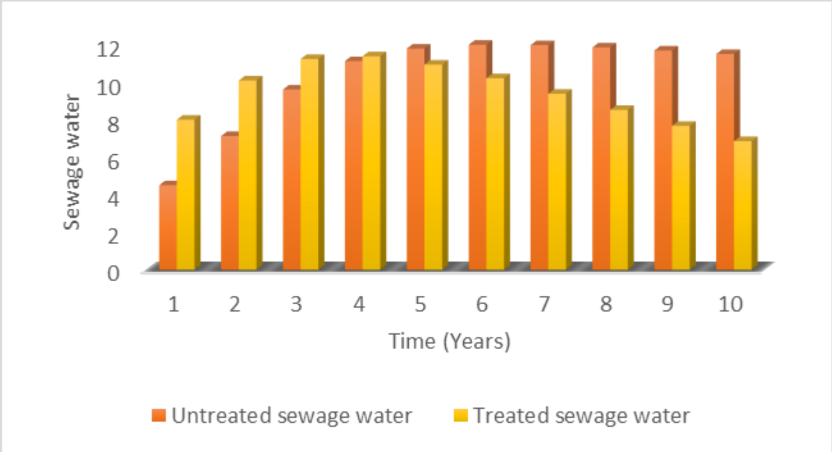

Figure 5. Volume of treated and untreated sewage water

Figure 5 shows the cubic volume of sewage water that can be either treatable or untreatable. They work proportionally. It means untreated sewage water increases with the time and treated sewage water decreases with the time. Which indicates that government should put control on untreated sewage water produced by household and industry. Next figure shows how much control is required.

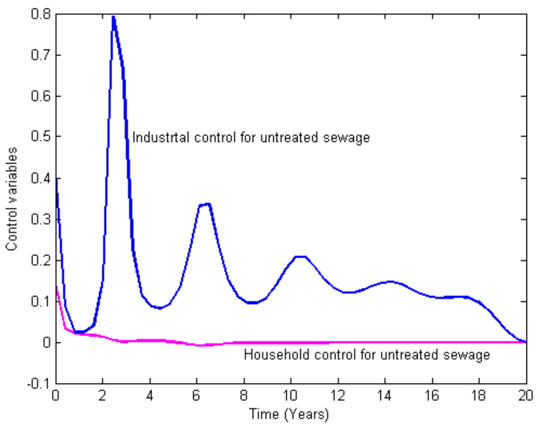

Figure 6. Control variable

From figure 6, it can be easily determined that we should put control on industry more than household which produces untreated sewage water. We should give 13% control on household at initial stage. Then it can be decreased and get stabilized. But control for industry is fluctuating. We have to give 80% control after 2.5 years which is the highest and then after 20 years no control will be required for some small-time span.

VI. CONCLUSION

In this paper, a mathematical model of recycling of sewage water is developed to reuse the treated sewage water. A mathematical model for recycling of sewage water is to study how we can use more reclaimed water instead of clean water. Recycling of sewage water is very beneficial. It can reduce and prevent the pollution, increases water availability, etc. These are the key motivators for executing reuse programmes. For this, government should construct more number of water recycling plants. We should also try to avoid producing untreated sewage water.

Using the parametric values given in the table, the basic reproduction number is 0.7398 which suggests that is 15% of sewage is treated then we can use 73.98% of reclaimed water.

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REFERENCES

[1]. N. Xiang, J. Sha, et al. Dynamic modeling and simulation of water environment management with a focus on water recycling. Water, 6:17-31. 2013.

[2]. M. V. Roon. Water localisation and reclamation: Steps towards low impact urban design and development. *Journal of environmental management*, 83:437-447, 2007.
[3]. S. Toze. Reuse of effluent water—benefits and risks. *Agricultural water management*, 80:147-159, 2006.
[4]. C. H. Huang, C. T. Chamg, et al. A mathematical programming model for water usage and treatment network design. *Industrial & Engineering Chemistry Research*, 38:2666-2679, 1999.
[5]. N. Haruvy. Agricultural reuse of wastewater: nation-wide cost-benefit analysis. *Agriculture, Ecosystems & Environment*, 66:113-119, 1997.
[6]. M. Y. Sklarz, A. Gross, et al. Mathematical model for analysis of recirculating vertical flow constructed wetlands. Volume 44. *Water research*, 2010.
[7]. K. V. Gernaey, M. C. Van, et al. Activated sludge wastewater treatment plant modelling and simulation: state of the art. *Environmental Modelling & Software*, 19:763-783, 2004.
[8]. N. Pochai, S. Nopparat, et al. A mathematical model of water pollution control using the finite element method. *PAMM*, 6:755-756, 2006.
[9]. R. Kaushik. Mathematical modelling on water pollution and self-purification of river Ganges, 2015.
[10]. D. G. DeCoursey. Mathematical models for nonpoint water pollution control. *Journal of Soil and Water Conservation*, 40:408-413, 1985.
[11]. N. H. Shah, M. H. Satia, et al. Optimum Control for Spread of Pollutants through Forest Resources. *Applied Mathematics*, 8:607-620, 2017.
[12]. N. H. Shah, M. H. Satia, et al. Optimal Control on depletion of Green Belt due to Industries. *Advances in Dynamical Systems and Applications*, 12:217-232, 2017.
[13]. W.H. Fleming and R. W. Rishel. Deterministic and stochastic optimal control. *Springer Science & Business Media*, 2012.
[14]. L.S. Pontriagin, V.G. Boltyanskii, *et al*. The Mathematical Theory of Optimal Process. *Gordon and Breach Science Publishers*, 1986.

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