ISSN (e): 2250 – 3005 || Volume, 07 || Issue, 06|| June – 2017 ||
International Journal of Computational Engineering Research (IJCER)

Wave-Current Interaction Model on an Exponential Profile

Dr. Evans F. Osaisai,
Department of Mathematics, Niger Delta University, PO Box 1693, Yenagoa, Bayelsa State

ABSTRACT
We develop a model that approximates the exponential depth, which exhibits the behavior of linear depth particularly in the surf zone. The main effect of the present exponential depth is found in the shoaling zone, where the depth remains finite. The basic description and the outcome is essentially rip currents where in the surf zone the wave behavior is the same as found in the linear depth case. In the shoaling zone the present exponential depth exhibits the hypergeometric functions.
Keywords: Surf zone, Shoaling zone, wave setup, wave setdown, rip currents, radiation stress, wave action.

I. INTRODUCTION

Wave transformation in the surf zone is the dominant factor in the hydrodynamics of the nearshore circulation, and consequent sediment transport. A global description in terms of the spatial variation of such quantities such as wave action, wave action flux and wave radiation stress are the driving entities we have used to describe the generation by waves of mean currents in the nearshore zone. Studies on the interaction between waves and currents span nearly half a century. Mainly driven by a combination of engineering, shipping and coastal interests, there has been much research on shoaling nonlinear waves, on how currents affect waves and how waves can drive currents. This last aspect is the main concern in this paper.

The basis for this subject was laid down by Longuet-Higgins & Stewart(1960, 1961), who analyzed the nonlinear interaction between short waves and long waves (or currents), and showed that variations in the energy of the short waves correspond to work done by the long waves against the *radiation stress* of the short waves. In the shoaling zone this radiation stress leads to what are now known as wave setup and wave setdown, surf beats, the generation of smaller waves by longer waves, and the steepening of waves on adverse currents, see Longuet-Higgins & Stewart (1962, 1964). The divergence of the radiation stress was shown to generate an alongshore current by obliquely incident waves on a beach (Longuet-Higgins 1970).

The action of shoaling waves, and wave breaking in the surf zone, in generating a wave-generated mean sea-level is well-known and has been extensively studied, see for instance the monographs of Mei (1983) and Svendsen (2006). The simplest model is obtained by averaging the oscillatory wave field over the wave phase to obtain a set of equations describing the evolution of the mean fields in the shoaling zone based on small-amplitude wave theory and then combining these with averaged mass and momentum equations in the surf zone, where empirical formulae are used for the breaking waves. These lead to a prediction of steady set-down in the shoaling zone, and a set-up in the surf zone. This agrees quite well with experiments and observations, see Bowen et al (1968) for instance. However, these models assume that the sea bottom is rigid, and ignore the possible effects of sand transport by the wave currents, and the wave-generated mean currents. Hydrodynamic flow regimes where the mean currents essentially form one or more circulation cells are known as rip currents. These form due to forcing by longshore variability in the incident wave field, or the effect of longshore variability in the bottom topography (Kennedy 2003, 2005,Yu & Slinn 2003, Yu 2006 and others). They are often associated with significant bottom sediment transport, and are dangerous features on many surf beaches (Lascody 1998 & Kennedy 2005).

There is a vast literature on rip current due to wave-current interactions, see the recent works by (Horikawa 1978, Damgaard *et al.* 2002, Ozkan-Haller & Kirby 2003,Yu & Slinn 2003, Yu 2006, Falques, Calvete & Monototo 1998 a and Falques *et al* 1999b, Zhang *et al* 2004 and others) and the references therein. Our purpose in this paper is to extend the wave-current interaction model examined in [29, 30] to the exponential depth.

Essentially we derive a model for the interaction between waves and currents. The aim is to provide analytical solution for waves in the nearshore zone on time scales longer than an individual wave. This is possible on long-time scales using the wave-averaging procedure often employed in the literature (see the textbook by Mei, 1983). We describe solutions for rip currents, in the shoaling zone matched to the surf zone, for two different beach profiles.

The structure of the mathematical model is based on the Euler equations for an inviscid incompressible fluid. We then employ an averaging over the phase of the waves, exploiting the difference in time scales between the waves and the mean flow, which is our main interest. The nearshore zone is divided into regions, a shoaling

zone where the wave field can be described by linear sinusoidal waves, and the surf zone, where the breaking waves are modelled empirically. The breakerline is fixed at $x = x_b$ but in general could vary.
In the shoaling zone wave field, we use an equation set consisting of a wave action equation, combined with the local dispersion relation and the wave kinematic equation for conservation of waves. The mean flow field is then obtained from a conservation of mass equation for the mean flow, and a momentum equation for the mean flow driven by the wave radiation stress tensor. In the surf zone, we use a standard empirical formula for the breaking wave field, together with the same mean flow equations.
The mathematical formulation and derivation are given in details see [29, 30, 35] where we use the usual wave-averaged mean field equations that are commonly used in the literature. In section 2 we introduce a description of the rip current formation and examine the consequences for both shoaling and surf zones. Then in section 3 we employ section 2 to extend our previous results see Osaisai, E. F (2013), now to include the rip currents behavior on an exponential depth profile. We conclude with a discussion in section 4.

II. RIP CURRENTS

Rip currents are essentially regimes where the mean currents form one or more circulation cells. These form due to forcing by longshore variability in the incident wave field, or the effect of longshore variability in the bottom topography (Kennedy 2003, 2005,Yu & Slinn 2003, Yu 2006 and others). They are often associated with significant bottom sediment transport, and are dangerous features on many surf beaches (Lascody 1998 & Kennedy 2005). Here we consider a steady-state model driven by an incident wave field which has an imposed longshore variability. The wave field satisfies the wave action equation which in the present steady-state case reduces to

$$\frac{\partial}{\partial x}(Ec_g\cos\theta) + \frac{\partial}{\partial y}(Ec_g\sin\theta) = 0. \qquad (1)$$

Here we again assume that $h = h(x)$ and that consequently the frequency ω and the longshore wavenumber l are constants, while the onshore wavenumber K is then determined from the dispersion relation. We then suppose that the wave energy E has the form

$$E = E_0(x)\cos Ky + F_0(x)\sin Ky + G_0(x), \qquad (2)$$

where the longshore period $2\pi/K$ is imposed.

Once the expression (2) has been determined, we may then properly evaluate to obtain the radiation stress fields. Our aim here then is to describe how steady-state rip currents are forced by this longshore modulation of the incident wave field, especially in the surf zone. We note that one problem in the modeling of rip currents is that the rip current length scales may be comparable with the length of the incident waves so that the applicability of the present averaging approach is in doubt Peregrine (1998). Again in almost all wave averaging approaches there has been a simplification of the shoreline conditions by not taking into account the motion of the instantaneous shoreline up and down the beach Peregrine (1998). Nevertheless, the forced two-dimensional shallow water equations that we use here are characteristic of many nearshore studies (Horikawa 1978, Damgaard *et al.* 2002 , Ozkan-Haller & Kirby 2003,Yu & Slinn 2003, Yu 2006, Falques, Calvete & Monototo 1998a and Falques *et al* 1999b and others). Then, omitting the overbars as before, the mean momentum equations [**??**] in the present steady-state case reduce to

$$H[U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y}] = -g\,H\frac{\partial \zeta}{\partial x} - [\tau_x], \qquad (3\text{ a,b})$$

$$H[U\frac{\partial V}{\partial x} + V\frac{\partial V}{\partial y}] = -g\,H\frac{\partial \zeta}{\partial y} - [\tau_y],$$

where the stress terms define are

$$\tau_x = \frac{\partial S_{11}}{\partial x} + \frac{\partial S_{12}}{\partial y} \text{ and } \tau_y = \frac{\partial S_{21}}{\partial x} + \frac{\partial S_{22}}{\partial y}. \qquad (4)$$

Next we observe that the steady-state mean mass equation can be solved using a transport stream function $\psi(x,y)$, that is

$$U = -\frac{1}{H}\frac{\partial \psi}{\partial y} \quad \text{and } V = \frac{1}{H}\frac{\partial \psi}{\partial x}, \qquad (5)$$

Next, eliminating the pressure, we get the mean vorticity equation

$$\psi_x(\frac{\Omega}{H})_y - \psi_y(\frac{\Omega}{H})_x = [\frac{\tau_x}{H}]_y - [\frac{\tau_y}{H}]_x \qquad (6)$$

where Ω is define as

$$\Omega = V_x - U_y = (\frac{\psi_x}{H})_x + (\frac{\psi_y}{H})_y. \qquad (7)$$

We shall solve this equation (6) in the shoaling zone $x > x_b$ and in the surf zone $x < x_b$, where as before $x = x_b$ is the fixed breaker line. It will turn out that the wave forcing occurs only in the surf zone, but continuity implies that the currents generated in the surf zone must be continued into the shoaling zone.

2.1 SHOALING ZONE

In $x > x_b$ we shall assume that $H \approx h$ as ζ is $O(a^2)$. Then we shall use the expressions [**??**,**??**] to evaluate the radiation stress tensor. For simplicity, we shall also use the shallow-water approximation that $c_g \approx c \approx gh$, and so we get that

$$S_{11} = E(\cos^2\theta + \frac{1}{2}), S_{12} = S_{21} = E\sin\theta\cos\theta, S_{22} = E(\sin^2\theta + \frac{1}{2}) \qquad (8)$$

These expressions are in principal known at this stage, and so we can proceed to evaluate the forcing term on the right-hand side of (6). To assist with this we recall Snell's law

$$\sin\theta = \frac{\sqrt{h}}{\sqrt{h_b}}\sin\theta_b$$

where h_b and θ_b are the water depth and incidence angle at the breaker-line. Now the energy equation (1) has the approximate form

$(E\, c \cos\theta)_x + (E\, c \sin\theta)_y = 0$,

and using Snell's law, this can be written as

$$(E\cos^2\theta)_x + (E\sin\theta\cos\theta)_y + E\frac{c_x}{c} = 0,$$

and so $$\tau_x = \frac{1}{2}E_x - E\frac{c_x}{c}.$$

We can also deduce from (1) that

$(E \sin\theta \cos\theta)_x + (E \sin^2\theta)_y = 0$,

and so $$\tau_y = \frac{1}{2}E_y.$$

We can now evaluate the right-hand side of (6), and find that its identically zero,

$$[\frac{\tau_x}{h}]_y - [\frac{\tau_y}{h}]_x = 0.$$

Thus in the shoaling zone there is no wave forcing in the mean vorticity equation, although of course there will be a mean pressure gradient. However, this does not concern us since here our aim is to find only the flow field. Note that the result that there is no wave forcing in the vorticity equation does *not* need the specific form (2), and is based solely on the steady-state wave energy equation (1). The specific form (2) is only used in the surf zone.

With no forcing term, the vorticity equation (6) can be solved in the compact form, noting that we again approximate H with h,

$$\frac{\Omega}{h} = F(\psi). \qquad (9)$$

But here $F(\psi) = 0$ from the boundary conditions in the deep water as $x \to \infty$, where the flow field is zero. Thus our rip current model has zero vorticity in the shoaling zone. It follows that we must solve the equation

$$\Omega = (\frac{1}{h}\psi_x)_x + (\frac{1}{h}\psi_y)_y = 0, \qquad (10)$$

in $x > x_b$. Since $h = h(x)$ we can seek solutions in the separated form

$$\psi = X(x)Y(y) \qquad (11)$$

with the outcome that

$$(\frac{X_x}{h})_x - \frac{K^2 X}{h} = 0, \quad Y_{yy} + K^2 Y = 0 \qquad .(12)$$

We note the separation constant $K^2 = 2\pi/L$ must not be zero, and is in fact chosen to be consistent with the modulation wavenumber of the wave forcing. Without loss of generality, we can choose,

$$Y = \sin Ky. \qquad (13)$$

For each specific choice of $h(x)$ we must then solve for $X(x)$ in $x > x_b$, with the boundary condition that $X \to 0$ as $x \to \infty$. We shall give details in the following subsections. Otherwise we complete the solution by solving the system (6) in the surf zone, and matching the solutions at the breakerline, $x = x_b$ where the streamfunction ψ must be continuous, and in order to have a continuous velocity field we must also have that ψ_x is continuous.

2.2 SURF ZONE

The waves become steeper as they propagate into shallow water because the group velocity decreases. To maintain the wave energy flux, wave amplitude increases towards the surf zone. But in the surf zone, we shall here assume that the expression (2) holds, where the functions $E_0(x)$,$F_0(x)$,$G_0(x)$ are determined empirically, as in Chapter 3. To determine the wave forcing term in the mean vorticity equation (6) in the surf zone $x < x_b$ we shall assume that $\theta = \theta_b << 1$ so that, on using (4) and (8) we get that here

$$\tau_x = \frac{3}{2}E_x\,, \quad \tau_y = \frac{1}{2}E_y.$$

Hence (6) now becomes, where we again approximate H with $h(x)$,

$$\frac{\partial\psi}{\partial x}\frac{\partial\tilde{\Omega}}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial\tilde{\Omega}}{\partial x} = \frac{E_{xy}}{h} + \frac{E_y h_x}{2h^2} = \frac{(h^{1/2}E_y)_x}{h^{3/2}}, \qquad (14)$$

where here $\tilde{\Omega} = \Omega/h$ is the potential vorticity. Since here the wave forcing is given by (2), that is

$$E = E_0(x)\cos Ky + F_0(x)\sin Ky + G_0(x), \qquad (15)$$

then the right-hand side of equation (14) is of the form

$$(K\cos\,Ky)\frac{(h^{1/2}F_0)_x}{h^{3/2}} - (K\sin\,Ky)\frac{(h^{1/2}E_0)_x}{h^{3/2}} \qquad (16)$$

Note that the unmodulated term $G_0(x)$ plays no role here at all, although of course it will contribute to the wave setup. In order to match at $x = x_b$ with the expression (13) for the streamfunction in the shoaling zone, we should try for a solution of (14) of the form

$$\psi = F(x)\sin Ky + G(x), \qquad \text{in } x < x_b. \qquad (17)$$

The matching conditions for the streamfunction and velocity field at the breakerline $x = x_b$ require that

$F(x_b) = X(x_b)$, $F_x(x_b) = X_x(x_b)$, $G(x = x_b) = 0$, $G_x(x_b) = 0$.

The expression (17) yields

$$\Omega = \tilde{F}\sin Ky + \tilde{G} \qquad (18)$$

where $\tilde{F}$ and $\tilde{G}$ are differential operators and defined respectively as

$$\tilde{F} = (\frac{F_x}{h})_x - \frac{K^2F}{h} \qquad (19)$$

$$\tilde{G} = Z_x,\;\; Z = \frac{G_x}{h} \qquad (20)$$

The left-hand side of equation (14) is of the form,

$$\psi_x(\frac{\Omega}{h})_y - \psi_y(\frac{\Omega}{h})_x = K\sin\,Ky\,\cos\,Ky[F_x\frac{\tilde{F}}{h} - F(\frac{\tilde{F}}{h})_x + K\cos\,Ky[G_x\frac{\tilde{F}}{h} - F(\frac{\tilde{G}}{h})_x]\,.(21)$$

We can now equate the two expressions (16) and (21) to get that

$$F_x\frac{\tilde{F}}{h} - F(\frac{\tilde{F}}{h})_x = 0\,, \qquad (22a,b,c)$$

$$(G\frac{\tilde{F}}{h} - F(\frac{\tilde{G}}{h})_x) = \frac{(h^{1/2}F_0)_x}{h^{3/2}}$$

$$(\frac{h^{1/2}E_0)_x}{h^{3/2}} = 0\,.$$

These equations determine the rip-current flow field in the surf zone. The last equation (22c) gives that $E_0 \sim 1/h^{1/2}$, which is an unacceptable singularity as $h \to 0$. Hence we must infer that in the surf zone at least, $E_0 = 0$. Next we deduce from equation (22a) that

$$\frac{\tilde{F}}{h} = CF \text{ where } C \text{ is a constant,} \qquad (23a)$$

and then (22b) yields that

$$F(CG - (\frac{\tilde{G}}{h})_x) = \frac{(h^{1/2}F_0)_x}{h^{3/2}} \qquad (23b)$$

The boundary conditions at $x = 0$ where $h = 0$ are that both mass transport fields U,V should vanish, that is from (5) ψ = constant and $\psi_x/h = 0$, which implies that

$$F = F_x = 0,\; G = \text{constant},\; \frac{G_x}{h} = 0\,,\;\text{at } x = 0. \qquad (24)$$

As above there are also the matching conditions for both F and G separately at the breakerline, that is for F we have that

$$\frac{F_x(x_b)}{F(x_b)} = \frac{X_x(x_b)}{X(x_b)}\,,\;\text{at } x = x_b$$

where we note that here the right-hand side is a known quantity, depending only on K and x_b. Next we see that equation (23a) reduces to

$$(\frac{F_x}{h})_x - \frac{K^2 F}{h} = ChF \quad (25)$$

Together with the boundary conditions at $x = 0, x = x_b$ this is essentially an eigenvalue problem for $F(x)$ with eigenvalue C. In general it is solved approximately since we shall assume that $Kx_b << 1$. Once $F(x)$ is known we can solve (23b), together with the appropriate boundary conditions to get $G(x)$ to complete the solution.

III. APPLICATION TO THE EXPONENTIAL DEPTH

3.1 IN THE SHOALING ZONE

Here equation (25) admits the profile, $h(x) = d(1 - e^{-\xi x})$. This is similar to the linear depth profile in the nearshore zone, but as $x \to \infty$, $h \to d$, which is finite. Thus in the shoaling zone far offshore, the bottom is flat. We put $u = e^{-\xi x}$ and then equation (12) becomes

$$u^2 X_{uu} + \frac{u}{1-u} X_u - L^2 X = 0, \quad L^2 = \frac{m^2}{\xi^2} \quad (26)$$

Now let $X = u^m Y$ to get

$u(1 - u)\, Y_{uu} + (2m + 1 - 2mu)Y_u + mY = 0$, where $L^2 = m^2$ hence $m_1 = L$ and $m_2 = -L$.

This has hypergeometric functions as solutions

F(a,b,c;u) where $m = -ab$, $2m = a + b + 1$ and $c = 2m + 1$, so that $c = a + b$.

The general solution of the hypergeometric equation has the form

$Y = C_1 F(a,b,a + b + 2;u) + C_2 u^{-2m} F(a,b,1;u)$.

The boundary condition at $u = 0, x \to \infty$ gives $C_2 = 0$ so the second solution is not present. Hence we get

$X = C_1 u^m [F(a,b,a + b + 2;u)$. (27)

3.2 IN THE SURF ZONE

Similarly with $u = e^{-\xi x}$, equation (25) becomes

$$u^2 F_{uu} + \frac{u}{1-u} F_u - L^2 F = \frac{Cd^2}{\xi^2}(1-u)^2 F. \quad (28)$$

We note that the shoreline boundary condition is $X = 0$ at $x = 0$ when $u = 1$. At the breakerline where $x = x_b$ then $X = u^m F(a,b,a + b + 2,u)$ and $X_x = \xi\,(-uX_u)$ and now one gets

$$\frac{xX_x}{X} = -\xi\, x \frac{uX_u}{X},$$

Where $$\frac{uX_u}{X} = m + u\frac{F_u}{F}.$$

The corresponding hypergeometric function, as given by Abramowitz, M. & Stegun, I. A (1964) is

$$F(a,b,a+b+2;u) = \frac{\Gamma(2)\Gamma(a+b+2)}{\Gamma(a+2)\Gamma(b+2)} \sum_{n=0}^{1} \frac{(a)_n (b)_n}{n!(1-2)_n}(1-u)^n - \frac{\Gamma(a+b+2)}{\Gamma(a)\Gamma(b)}(u-1)^2$$
$$\sum_{n=0}^{\infty} \frac{(a+2)_n (b+2)_n}{n!(n+2)!}(1-u)^n [\ln(1-u) - \psi(n+1) - \psi(n+3) + \psi(a+n+2) + \psi(b+n+2)].$$

As $x \to 0$, $1 - u \to \alpha x$ and so F has the structure of a power series in

$(1 - u = \xi x)$, with some multiplicative log terms. To leading order exactly like the linear depth case, but we need to find the corresponding constants. Thus we get that

$$C_0 = \frac{\Gamma(2L+1)}{\Gamma(a+2)\Gamma(b+2)}, \quad C_1 = -abC_0 = mC_0; \quad C_2 = \frac{\Gamma(2L+1)}{\Gamma(a)\Gamma(b)} = -m^2 C_0 \text{ ,and}$$

$F(a,b,a + b + 2;u) \approx C_0 + C_1(1 - u) + C_2(1 - u)^2 \ln(1 - u)$.

Using these approximations, we finally find that near $x = x_b$ (but $x > x_b$) , for $Kx_b << 1$,

$$\frac{xX_x}{X} = 2L^2 x^2 \ln x .$$

To leading order this agrees exactly with the corresponding result from the linear depth case. It follows that in the surf zone, where the depth profile is effectively linear, the solution for λ and $F(x)$ will be the same as that in found in [29, 30]. Thus the main effect of this present exponential depth profile is found only in the shoaling zone where here the depth remain finite as $x \to \infty$. Thus we get from (11, 13) in $x > x_b$ the normalized stream function ψ_n is given by

$$\psi_n = \frac{X(x)}{X(x_b)} \sin(Ky), \text{for} \quad x > x_b, \quad (29)$$

The two plots in figure [1] for linear and exponential depth respectively in the shoaling zone. For the case shown for d = 10m, we see that there is virtually no difference. However for a depth d = 4.0m there is some noticeable difference, in that for the exponential depth profile, the vortex centre is further offshore.

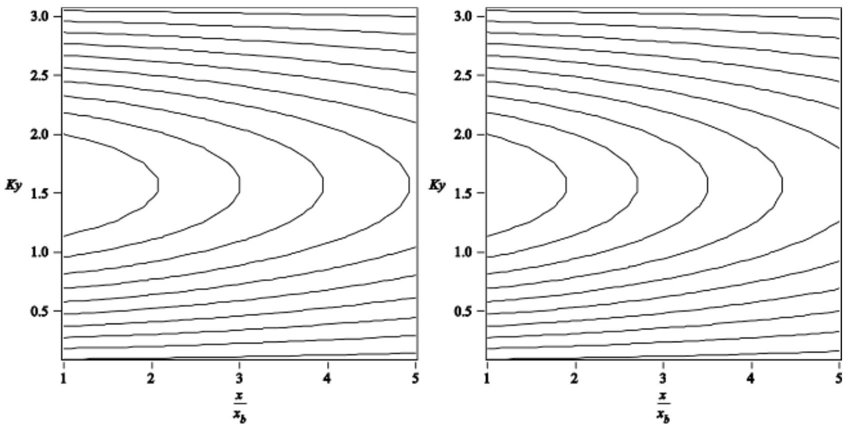

Figure 1: Plots of rip currents in the shoaling zone for linear and exponential depth profile respectively for a depth d = 10m where $Kx_b = 0.2$, a depth at the breaker line of $\xi x_b = 0.2$; note that then $L = 1.0$.

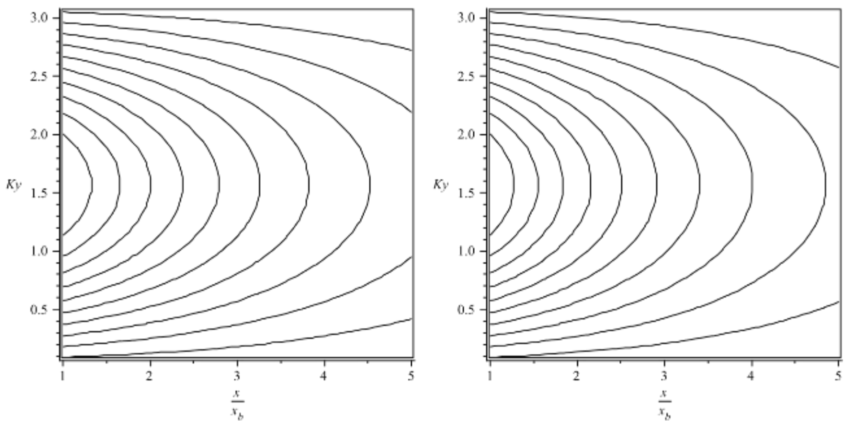

Figure 2: Plots of rip currents in the shoaling zone for linear and exponential depth profile respectively for a depth d = 4.0m where $Kx_b = 0.2$, a depth at the breaker line of $\xi x_b = 0.5$; note that then $L = 1.0$.

Next for the y-independent component, we need to solve for $G(x)$ from (23b). As above, we approximate $F = A_0x^2$, and we use the empirical expression $F_0 = \gamma^2h^2/8$.

$$Z_{xx} - \frac{1}{x}Z_x - \alpha^2x^2CG = -\frac{5g\alpha^2\gamma^2}{16A_0x} \qquad (30)$$

There is a singularity at $x = 0$, which can be analyzed by setting $u = x^2$ so that (30) becomes

$$4Z_{uu} - \lambda Z = -\frac{5g\alpha^3\gamma^2}{16A_0u^{3/2}}\,, \qquad (31)$$

and $\lambda = C\alpha^2$ as before. For the particular solution, we can balance the dominant term Z_{uu} on the left hand side with the forcing term to get that $Z \approx$ constant $u^{1/2}$, and so we get that the particular solution is

$Z_p = \frac{5g\,\alpha^2\gamma^2 x}{12\,A_0}$.

Note that this is a smooth function as $x \to 0$. Next the homogeneous solutions are, since we know that $\lambda < 0$

$$Z_{\square} = C_2\sin[\left(\frac{\sqrt{\tilde{\lambda}u}}{2}\right) + C_3\cos(\frac{\sqrt{\tilde{\lambda}u}}{2})]$$

where we have put $\tilde{\lambda} = -\lambda > 0$ and the full solution is

$Z = Z_{\square} + Z_p$.

When $u = 0$ we must have $Z = 0$, which requires that $C_3 = 0$. Also we can impose the condition that Z =0 at $x = x_b$ and so we finally get

$$Z = \frac{5g\alpha^2\gamma^2 x_b}{12A_0}\left(\frac{x}{x_b} - \frac{1}{\sin\sqrt{2}}\sin\left[\frac{\sqrt{2}x^2}{x_b^2}\right]\right) \quad (32)$$

Finally the complete the stream function $G(x)$ is found by integrating $G_x = hZ$ subject to the boundary condition that $G = 0$ at $x = x_b$. Thus we get that, for $0 < x < x_b$,

$$G(x) = \frac{5g\alpha^3\gamma^2 x_b^3}{12A_0}\left(\frac{x^3}{3x_b^3} + \frac{1}{2\sqrt{2}\sin\sqrt{2}}\cos\left[\frac{\sqrt{2}x^2}{x_b^2}\right] - \frac{1}{3} - \frac{1}{2\sqrt{2}\tan\sqrt{2}}\right) \quad (33)$$

Note in particular that $G(0) \neq 0$ and is the net mean longshore mass transport in the rip current system.

The combined expressions (**27**, **29**, 33) complete the solution, where we recall that the constant C is given by (**??**) (since $\lambda = C\alpha^2$), or their respective higher-order corrections. Note that the amplitude of $F(x)$ at $x = x_b$ is given by

$F(x_b) = C_1 x_b K_1(Kx_b)$.

On using the approximation $Kx_b << 1$, and the approximate expression (**??**), this reduces to

$$F(x_b) = \frac{2A_0 x_b^2}{3} = \frac{C_1}{K}.$$

The rip-current system contains a free parameter A_0 or its equivalent. We choose to define this free parameter to be the value of $F(x_b)$ and normalize the full solution by this value. Thus we get from (11, 13) in $x > x_b$, and (17 , **??**, 33) in $x < x_b$ that the normalized streamfunction ψ_n is given by

$$\psi_n = \frac{X(x)}{X(x_b)}\sin(Ky), \text{for} \quad x > x_b, \quad (34)$$

$$\psi_n = \frac{F(x)}{F(x_b)}\sin Ky + R\frac{G(x)}{G(0)}, \quad \text{for } 0 < x < x_b. \quad (35)$$

Here $R = G(0)/F(x_b)$ is a free parameter. Thus a larger (smaller) R decreases the circulation of the rip-current system *vis-a-vis* that of the longshore current component. From (**??**, 33) we find that here

$$R = \frac{5g\alpha^3\gamma^2 x_b}{8A_0^2}\left(-\frac{1}{3} - \frac{1}{2\sqrt{2}\tan\sqrt{2}}\right). \quad (36)$$

Note that with all other parameters fixed, a larger (smaller) slope α increases (decreases) R. In order to estimate typical values for R we note that from (**??**) the longshore velocity field in the "sin(Ky)"-component scales as $V_c = A_0/\alpha$, while the longshore component then scales with RV_c. Taking account of the actual numerical values in the expressions given above, we find that a suitable values are $R \approx -0.1$. Plots of ψ_n are shown in figures 3, 4, **??** for $R = -0.02, -0.1, -0.2, -0.5$ respectively, with $Kx_b = 0.2$ (noting that our present theory requires that Kx_b is small).

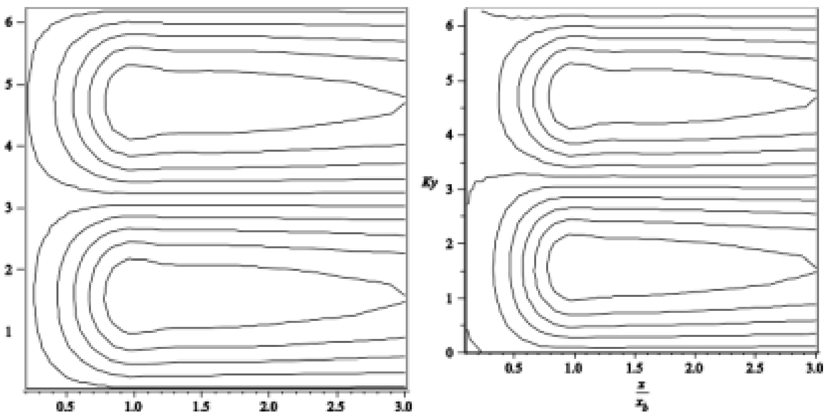

Figure 3: Plot of the rip current streamlines for a linear depth profile, given by equation (35) where $F(x)$ and $G(x)$ are equations (**??**) and (33) respectively for $R = -0.02$.

From the plots in figures 3, 4, **??** we see that as $|R|$ increases, the core of the rip current circulation moves from the shoaling zone towards the surf zone. The reason for this is that the solution we have constructed is essentially a free vortex defined by $X(x)\sin Ky$ in the shoaling zone $x > x_b$, and $F(x)\sin Ky$ in the surf zone $x < x_b$, perturbed by a longshore component $G(x)$ in the surf zone. Since $R < 0$, this longshore component opposes the vortex flow in the cell $0 < Ky < \pi$ but is in sympathy for $\pi < Ky < 2\pi$. This has the effect of moving the vortex cell further offshore in the sector $\pi < Ky < 2\pi$ relative to the sector $0 < Ky < \pi$. Note that $|R|$ increases as the wave forcing increases, or as the slope α increases, or as the depth αx_b at the breaker line increases.

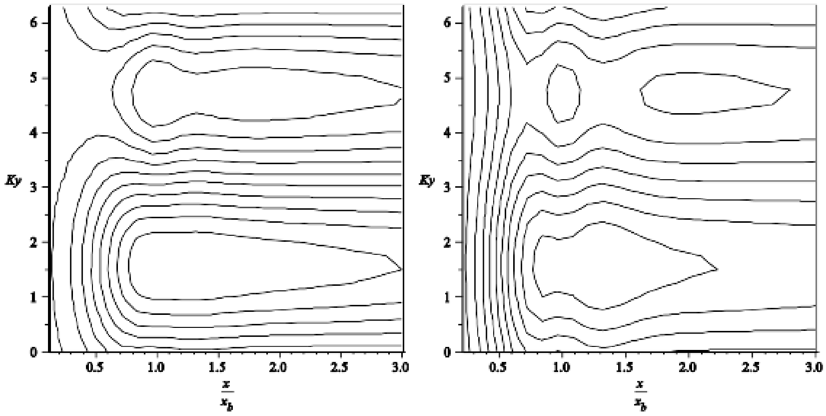

Figure 4: As for figure 3 but $R = -0.1$ and $R = -0.5$ respectively.

IV. WAVE SETUP

4.1 SHOALING ZONE

Once the velocity field has been determined, we can return to the momentum equation (3) to get the mean pressure field ζ. In the shoaling zone, the flow is irrotational and it is easily seen that

$$(g\zeta + \frac{U^2}{2} + \frac{V^2}{2})_x = -\frac{3E_x}{2h},$$

$$(g\zeta + \frac{U^2}{2} + \frac{V^2}{2})_y = -\frac{E_y}{2h},$$

Using the approximation that $Ec \approx$ constant in the shallow water, we find that

$$g\zeta + \frac{U^2}{2} + \frac{V^2}{2} = -\frac{E}{2h}, \quad x > x_b. \qquad (37)$$

Here a constant of integration has been set to zero, while U, V are given by (5) where ψ is given by (11, 13). Thus we get that

$$g\zeta = -\frac{K^2X^2}{2h^2}\cos^2 Ky - \frac{X_x^2}{2h^2}\sin^2 Ky + \frac{E}{2h}. \qquad (38)$$

We note that, unlike the velocity field, there is a y-independent component, a modulation with a wavenumber K and a superharmonic modulation with wavenumber $2K$.

4.2 SURF ZONE

Again using the components of (3) we now get that

$$(g\zeta + \frac{U^2}{2} + \frac{V^2}{2})_x - V\Omega = -\frac{3E_x}{2h},$$

$$(g\zeta + \frac{U^2}{2} + \frac{V^2}{2})_y + U\Omega = -\frac{E_y}{2h},$$

The first of these can then be integrated, using the expressions from the subsection 2.1, and assuming that in the surf zone $E \sim h^2$ as a function of x,

$$g\zeta + \frac{U^2}{2} + \frac{V^2}{2} = \frac{CF^2}{2}\sin^2 Ky + \frac{Z^2}{2} - \frac{3E}{h} + \sin Ky \int_0^x [\frac{F_x\tilde{G}}{h} + CFG_x]\,dx + f_1(y)\,.(39)$$

where $f_1(y)$ is an arbitrary function of y at this stage. Using the previous equations for F,G and the expressions for E the integral term can be simplified to give

$$\int_0^x [\frac{F_x\tilde{G}}{h} + CFG_x]\,dx = \frac{FZ_x}{h} + \int_0^x \frac{(h^{1/2}F_0)_x}{h^{3/2}}\,dx = \frac{5\gamma^2 h}{16}$$

Substituting, and assuming that in the surf zone the wave forcing is $E = F_0 \sim 1/8\gamma^2h^2$ we finally get that

$$g\zeta + \frac{U^2}{2} + \frac{V^2}{2} = \frac{CF^2}{2}\sin^2 Ky + \frac{FZ_x}{h}\sin Ky + \frac{Z^2}{2} - \frac{\gamma^2 h}{16}\sin Ky + f_1(y)\,.(40)$$

Next, differentiation of this expression with respect to y, and comparison with the y-component of the mean momentum equation shows that f_1 is a constant, showing that the Bernoulli expression on the left-hand side of (40) is a constant. This constant is just the pressure $g\zeta$ at $x = 0$, since all other terms in (40) vanish there, and so we can set this constant to zero, that is $f_1(y) = 0$ in (40).

Comparing the expressions for ζ as $x \rightarrow x_b$ from the shoaling zone (38) with that from the surf zone (40) we see that there must be a discontinuity in E across the breakerline $x = x_b$ in order to ensure the continuity of ζ. The reason for this is essentially because we imposed a discontinuity in the mean vorticity field across the fixed breakerline $x = x_b$ which is maintained by a discontinuity in the wave forcing, that is, the right-hand side of (6). Now we see that in addition to that there is discontinuity in E itself, given by

$$[E] = \{ChF^2 \sin^2 Ky + 2FZ_x \sin Ky + 5G_0(x)\}_{x=xb}. \quad (41)$$

Here $[\cdots]$ denotes the jump from above to below across $x = x_b$. Since the form of E in $x < x_b$ is already specified by (2) (with $E_0 = 0$), this expression implies that we must modify the expression for the wave field in $x > x_b$ according to (41). Fortunately this is allowed, since we saw earlier that we do not require the explicit form (2) for E in $x > x_b$; all that is required there is that E satisfy the wave energy equation (1). Hence we now see that the rip current system in fact modifies the wave field in $x > x_b$ through the term $E(x = x_b, y)$ deduced from (41). In particular we see that there is an additional modulation in $\sin Ky$ and a second harmonic $\cos 2Ky$, with amplitudes $2FZ_x(x = x_b)$ and $-ChF/2(x = x_b)$ respectively; the additional mean component can be accommodated by the choice of $G_0(x = x_b)$. For instance, for the linear depth profile these amplitudes are proportional to $-\gamma^2 h^2_b$ (independent of A_0) and $-\alpha^2 h_b A_0^2$ respectively; note that both of these amplitudes are negative.

V. CONCLUSION

We describe solutions for rip currents, in the shoaling zone matched to the surf zone, for two beach profiles. In the shoaling zone wave field, we use an equation set consisting of a wave action equation, combined with the local dispersion relation and the wave kinematic equation for conservation of waves. The mean flow field is then obtained from a conservation of mass equation for the mean flow, and a momentum equation for the mean flow driven by the wave radiation stress tensor. In the surf zone, we use a standard empirical formula for the breaking wave field, together with the same mean flow equations.

Some interesting remarks about the rip currents solution in these two flow regime were made. Firstly the rip current model in the shoaling zone has a zero vorticity. Thus the dynamics of the shoaling zone is only dependent on the state-state wave energy equation. Apparently this employs the method of separation of variables and superposition principle. Thus the solution we have constructed is essentially a free vortex defined $X(x)\sin Ky$. But in the surf zone is a form of wave forcing which sets the wave activities different from those of the shoaling zone. To determine wave forcing in the mean vorticity equation we assume that the wave angle becomes smaller. We also note here that the component of the radiation stress in the y momentum remains unchanged across the entire flow domain. This shows that it is only the x component of the radiation stress that play a leading role in the wave forcing. However, wave forcing encountered in the surf zone has an unmodulated term that *does not play a role* in the vorticity equation but only contribute to wave setup. To ensure continuity of the streamfunctions in the shoaling zone we match the solution at the breakerline by a matching condition with appropriate boundary conditions. Thus the rip currents solution in the surf zone is provided by the terms in the matching condition. The terms in the matching condition has a cross-shore width and a modulated longshore component. Therefore the rip currents solution in this regime due to wave forcing in the presence of rip cells can drive a longshore current. Nevertheless, the rip currents solution in the surf zone is also a free vortex defined by $F(x)\sin Ky$ but perturbed by a longshore component $G(x)$.

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