ISSN (e): 2250 – 3005 || Volume, 07 || Issue, 06|| June – 2017 ||
International Journal of Computational Engineering Research (IJCER)

The Effect of Bottom Sediment Transport on Wave Set-Up

Evans Osaisai
Department of Mathematics, Niger Delta University, PMB 071, AMASSOMA, NIGERIA

ABSTRACT
In this paper we augment the wave-averaged mean field equations commonly used to describe wave set-up and wave-induced mean currents in the near-shore zone, with an empirical sediment flux law depending only on the wave-induced mean current and mean total depth. This model allows the bottom to evolve slowly in time, and is used to examine how sediment transport affects wave set-up in the surf zone. We show that the mean bottom depth in the surf zone evolves according to a simple wave equation, whose solution predicts that the mean bottom depth decreases and the beach is replenished. Further, we show that if the sediment flux law also allows for a diffusive dependence on the beach slope then the simple wave equation is replaced by a nonlinear diffusion equation which allows a steady-state solution, the equilibrium beach profile.
Keywords: water waves, mean flow, sediment transport

I. INTRODUCTION

The action of shoaling waves, and wave breaking in the surf zone, in generating a wave-generated mean sea-level is well-known and has been extensively studied, see for instance the monographs of Mei (1983) and Svendsen (2006). The simplest model is obtained by averaging the oscillatory wave field over the wave phase to obtain a set of equations describing the evolution of the mean fields in the shoaling zone based on small-amplitude wave theory and then combining these with averaged mass and momentum equations in the surf zone, where empirical formulae are used for the breaking waves. These lead to a prediction of steady set-down in the shoaling zone, and a set-up in the surf zone. This agrees quite well with experiments and observations, see Bowen et al (1968) for instance. However, these models assume that the sea bottom is rigid, and ignore the possible effects of sand transport by the wave currents, and the wave-generated mean currents. Our purpose in this paper is to add an empirical model of sediment transport to the wave-averaged mean field equations and hence determine the effect of this extra term on wave set-up.

There is a vast literature on sediment transport due to waves, see the recent works by Caballeria *et al* (2002), Calvete *et al* (2001, 2002), Garnier *et al* (2006, 2008), Hancock et al (2008), Lane and Restrepo (2007), McCall *et al* (2010), Restrepo (2001), Restrepo and Bona (1995), Roelvink *et al* (2009) and Walgreen *et al* (2002) and the references therein. There are several methods to model the movement of bottom sediment by the combined action of waves and currents, and these can often be quite complicated, depending *inter alia* on the nature of the sediment, and whether the sediment is confined to the bottom boundary layer, or is suspended throughout a larger portion of the water column. Various models have been used to describe the formation of sand bars, ripples and sand waves, where it has usually been assumed that the wave field is quasi-periodic and non-breaking, see for instance the aforementioned articles and the review article by Blondeaux (2001). For the most part, application of these models to the near shore zone, where there is wave breaking, has been confined to numerical simulations. In particular, the effect of sediment transport on wave set-up, especially in the surf zone, does not appear to have been examined in analytical detail, which is contrast to the case when there is no sediment transport where a well-established analytical theory exists (see Mei (1983) or Svendsen (2006) for instance). To remedy this, we modify the well-known wave-averaged mean field equations by a bottom boundary condition that allows for the evolution of the bottom as sediment is moved. This leads to a single extra equation in the wave-averaged mean field model to represent the time evolution of the bottom, based on a relatively simple empirical law for the bottom sediment flux, based on the sediment transport models used in similar problems in the cited references above.

In section 2 we present the usual wave-averaged mean field equations that are commonly used in the literature, supplemented here by a bottom sediment transport term. We then examine the consequences for wave set-up in section 3. We conclude with a discussion in section 4.

II. FORMULATION

2.1 Wave field

In this section we recall the wave-averaged mean flow and wave action equations that are commonly used to describe the near-shore circulation (see Mei 1983 or Svendsen 2006 for instance). We suppose that the depth and the mean flow are slowly-varying compared to the waves. Then we define a wave-phase averaging operator $< f >= \bar{f}$, so that all quantities can be expressed as a mean field and a wave perturbation, denoted by a "tilde" overbar. For instance,

$\zeta = \bar{\zeta} + \tilde{\zeta}$. (1)

where ζ is the free surface elevation above the bottom located at $z = -h(\mathbf{x},t)$. Then outside the surf zone, the representation for slowly-varying, smallamplitude waves is, in standard notation,

$\tilde{\zeta}(\mathbf{x},t) \sim a\cos\theta$. (2)

Here $a = a(\mathbf{x}.t)$ is the wave amplitude and $\theta = \theta(\mathbf{x},t)$ is the phase, such that the wavenumber $\mathbf{k}$, frequency Ω are given by

$$\mathbf{k} = (k, l) = \nabla\theta, \quad \Omega = -\theta_t. \quad (3)$$

Here $\nabla = (\partial_x, \partial_y)$. The local dispersion relation is

$$\Omega = \omega + \mathbf{k}.\mathbf{U}, \quad \omega^2 = g\kappa\tanh\kappa H \text{ where } \kappa^2 = k^2 + l^2. \quad (4)$$

Here $\mathbf{U}(\mathbf{x},\mathbf{t})$ is the slowly-varying depth-averaged mean current (see below), and $H(\mathbf{x},t) = h(\mathbf{x},t) + \bar{\zeta}(\mathbf{x},t)$ is the total fluid depth, also a slowly varying function of $\mathbf{x}, t$.

The basic equations governing the wave field is then the kinematic equation for conservation of waves

$\mathbf{k_t} + \nabla\omega = 0$, (5)

which is obtained from (3) by cross-differentiation, the local dispersion relation (4), and the wave action equation for the wave amplitude

$A_t + \nabla\cdot(\mathbf{c}_g A) = 0$. (6)

Here $A = E/\omega$, where $E = ga^2/2$ is the wave energy per unit mass, and $\mathbf{c}_g = \nabla_{\mathbf{k}}\cdot\omega = \mathbf{U} + c_g\mathbf{k}/\kappa, (c_g = d\omega/d\kappa)$ is the group velocity.

2.2 Mean fields

The equations governing the mean fields are obtained by averaging the depthintegrated Euler equations over the wave phase. Thus the averaged equation for conservation of mass is

$H_t + \nabla\cdot(H\mathbf{U}) = 0$. (7)

For the velocity field we proceed in a slightly different way, that is we define

$\mathbf{u} = \mathbf{U} + \mathbf{u}'$, (8)

where we define $\mathbf{U}$ so that the mean momentum density is given by

$\mathbf{M} = H\mathbf{U} = \int_{-h}^{\zeta} < u dz >$ (9)

But now we need to note that $\mathbf{u}'$ does not necessarily have zero mean, and that $\mathbf{U}$ and $\bar{\mathbf{u}}$ are not necessarily the same. Indeed, from (8) and (9) we get that

$\bar{\mathbf{u}} = \mathbf{U} + < \mathbf{u}' >$, and $\int_{-h}^{\zeta} < u' dz > = 0$

However, $\mathbf{u}' = \tilde{\mathbf{u}} + O(a^2)$, so that $< \mathbf{u}' >$ is $O(a^2)$ and it follows that, correct to second order in wave amplitude,

$\mathbf{M} = H\bar{\mathbf{u}} + \mathbf{M}_w$, where $\mathbf{M}_w = -H < \mathbf{u}' > = < \zeta\,\tilde{\mathbf{u}}(x, 0, t) > = \frac{E}{\omega}\mathbf{k}$. (10)The term $\mathbf{M}_w$ in (10) is called the wave momentum, and can be expressed as $\mathbf{M}_w = H\mathbf{U}_s$ where $\mathbf{U}_s$ is the Stokes drift velocity. It follows that $\mathbf{U}$ is a Lagrangian mean flow.

Averaging the depth-integrated horizontal momentum equation yields (Mei 1983)

$(H\mathbf{U})_t + \nabla.(H\mathbf{U}\mathbf{U}) = -\nabla.\int_{-h}^{\zeta} < u'u' + pI dz > < + < p(z = -h) > \nabla h.$

An estimate of the bottom pressure term is made by averaging the vertical momentum equation to get

$$< p(z = -h) > - g(\bar{\zeta} + h) = \nabla. < \int_{-h}^{\zeta} w\,\mathbf{u}\,dz > + < \int_{-h}^{\zeta} w\,dz >_t.$$

For slowly-varying small-amplitude waves, the integral terms on the right hand side may be neglected, and so $< p(z = -h) > \approx g(\bar{\zeta} + h)$. Using this in the averaged horizontal momentum equation, and replacing the pressure p with the dynamic pressure $q = p + (z - \bar{\zeta})$ yields

$(H\mathbf{U})_t + \nabla.(H\mathbf{U}\mathbf{U}) = -\nabla.\mathbf{S} - gH\nabla\bar{\zeta}$ (11)

where $\mathbf{S} = < \int_{-h}^{\zeta} [\mathbf{uu} + q\mathbf{I}]dz > - < \frac{g}{2}\tilde{\zeta}^2 > \mathbf{I}$. (12)

Here $\mathbf{S}$ is the radiation stress tensor. In the absence of any basic background current, so that $\mathbf{U}$ is $O(a^2)$, we may use the linearized expressions (2, **??**) to find that

$\mathbf{S} \approx \mathbf{c}_g\mathbf{k}\frac{E}{\omega} + E[\frac{c_g}{c} - \frac{1}{2}]\mathbf{I}$. (13)

where the phase speed $c = \omega/\kappa$, correct to second order in the wave amplitude.

In summary, to this stage the wave field is described by equations (5, 6) for $\mathbf{k}, E$, while the mean field equations to be solved for the mean variables $U, \bar{\zeta}, \bar{H}$ are the averaged equation for conservation of mass (7) and the averaged equation for conservation of horizontal momentum (11), where the radiation stress tensor is given by (13). An additional equation is needed, and this is provided by the sediment transport flux law described in the next section 2.3.

2.3 Sediment flux law

To take account of bottom sediment transport, in addition to the kinematic bottom boundary condition,

$h_t + \mathbf{u}.\nabla h = -w$, at $z = -h(\mathbf{x}, t)$, (14)

a second bottom boundary condition is needed, which is an appropriate sediment flux law

$h_t = \nabla.\mathbf{Q}$,(15)

where $\mathbf{Q}$ is the sediment flux, evaluated at the bottom. The kinematic condition (14) has already been used in deriving the mean mass equation (7). Hence we now also average the sediment flux equation (15) so that

$\bar{h}_t = \nabla.\bar{\mathbf{Q}}$, (16)

where $\bar{\mathbf{Q}}$ is the wave-averaged sediment flux.

In the literature various flux laws have been proposed, depending on the assumed sediment type. Here we follow the type of formulation suitable for use, as here, with the wave-averaged field equations, see for instance, Caballeria *et al* (2002), Calvete *et al* (2001, 2002), Garnier *et al* (2006, 2008) , Lane and Restrepo (2007), McCall *et al* (2010), Restrepo (2001), Roelvink *et al* (2009) and Walgreen *et al* (2002), although there are subtle differences between the models used by these authors. Hence here we set

$\bar{\mathbf{Q}} = \frac{\mu}{1 - p_s}(\mathbf{Q}_b + \mathbf{Q}_s)$, (17)

where $p_s \approx 0.4$ is the bed porosity, and $\mu \approx 0.05$ is a measure of how often the waves are large enough to move the sediment. The quantities $\mathbf{Q}_{b,s}$ are the bed-load and suspended sediment fluxes respectively, and for wave-dominated situations are given by expressions of the form

$\mathbf{Q}_b = \nu_b(|\mathbf{u}_w|^2\mathbf{U} + \lambda_b|\mathbf{u}_w|^3\nabla b)$,

(18)

$\mathbf{Q}_s = \nu_s(H|\mathbf{u}_w|^3\,\mathbf{U} + \lambda_s|\mathbf{u}_w|^5\nabla b)$.

Here $|\mathbf{u}_w|$ is the wave velocity magnitude, and we recall from (10) that is the depth-averaged mean flow, while $b = \bar{h} - h_r(x)$ is the deviation of $\bar{h}$ from a reference depth $h_r(x)$, which can be taken to be either the initial depth, or an equilibrium depth. In each of these expressions, the second term proportional to ∇b is a diffusive term containing some explicit information about the effect of the beach slope on the sediment transport. We note that $\mathbf{U} = \bar{\mathbf{u}} + \mathbf{U}_S$ where $\mathbf{U}_s$ is the Stokes drift velocity, and especially in the surf zone, we will make the further approximation that the sediment is primarily moved by the waves, so that $\mathbf{U} \approx \mathbf{U}_s$. The coefficients ν_b and ν_s are for bed-load and suspended transport respectively, and representative values are $\nu_b = 1.8 \times 10^{-4}\, s^2\, m^{-1}$ and $\nu_s = 1.0 \times 10^{-3}\, s^3\, m^{-3}$, see the cited literature. These expressions are usually used outside the surf zone, and need modification inside the surf zone, where there are many various approaches, see the reviews by Roelvink and Broker (1993) and that in the Coastal Engineering Manual (2002). Here we assume that they remain valid, but only in qualitative form, inside the surf zone. Importantly, we remind that the formulation adopted here is for waveaveraged fields, and hence details of how the sediment transport is moved on each phase of the wave have been removed by the averaging process, although some relic of this survives in the diffusive terms. Equation (16) with the definitions (17, 18) complete the basic equation set.

III. WAVE SET-UP

3.1 Shoaling zone

To study wave set-up we first suppose that there is in the undisturbed state when there are no waves $h = h_0(x)$. We then make the simplification that there is no transverse dependence, so that all variables depend only on x,t. Thus, in particular, $\mathbf{k} = (k,0)$,$\mathbf{U} = (U,0)$. The shoaling zone is defined to be the region $x > x_b$, where x_b is defined below, and then the surf zone is $x_s < x < x_b$; here x_s is the shoreline where $H = 0$, and is found as part of the solution. This surf zone, where wave breaking occurs, is discussed in the next section 3.2 . Note that here we assume that the waves propagate in the negative x-direction so that $k < 0$, with $\Omega > 0$.

In the shoaling zone, we can assume that $\bar{\zeta}$, $\bar{U}$ are order a^2, so that $\Omega \approx \omega$ and $\bar{h} = h_0(x)$ to the same order of approximation. Then, to leading order we can seek steady solutions with no t-dependence for the wavenumber k and the wave energy E. As is well-known, then the equation for conservation of waves (5) shows that the frequency ω is a constant, and so k determined from the dispersion relation (4). As h decreases the waves refract towards the shore, and $|k|$ increases; in shallow water $|k| \sim \omega/(gh_0)^{1/2}$. The wave action equation (6) reduces to Ec_g is constant. Near the shore, where $c_g \approx (gh_0)^{1/2}$,

$$a^2 h_0^{1/2} \approx F_0 . \quad (19)$$

Note that if this is evaluated in shallow water, then $F_0 = a_\infty^2 h_\infty^{1/2}$, where a_∞ is the wave amplitude at a location offshore where $h_0 = h_\infty$. On the other hand, in deep water as $kh_0 \to \infty, c_g \to g/2\omega$, and then $F_0 = a_\infty^2 g/2\omega$, where a_∞ is the constant wave amplitude in deep water. The surf zone $x_s < x < x_b$ can now be defined by the criterion that h_b is that depth where $a/h_0 = A_c$, that is, $h_b^{5/2} = F_0/A_c^2$, defining an empirical breaking condition. A suitable value is $A_c = 0.44$, see Mei (1983) or Svendsen (2006).

Next, we examine the mean sediment flux term in (16) given by (17 , 18). In the shoaling zone $\mathbf{u}_w \approx \max|\tilde{\mathbf{u}}|$ where $\tilde{\mathbf{u}}$ is obtained from the usual linearised sinusoidal wave theory (see (2)). In shallow water $\tilde{u} \approx c\zeta/h$ and so it follows that $|\mathbf{u}_w|^2$ can be estimated as $ga^2/h = 2E/h$. Since in the present theory, the beach slope is assumed to be small, the diffusion terms in (18) will be omitted (but see section 3.4). It follows that the sediment flux (17) is expressed as

$$\bar{\mathbf{Q}}_{shoal} = \frac{2\mu E}{(1-p_s)h}(\nu_b + \nu_s(2Eh)^{1/2})\mathbf{U}. \qquad (20)$$

The Stokes drift is given by $\mathbf{U}_s = E\mathbf{k}/\omega h$, and is $O(E)$. Since we expect that the Eulerian mean flow $\bar{\mathbf{u}}$ forced by the waves will also be at least $O(E)$, it follows that $\mathbf{U}$ is $O(E)$. Hence, in the shoaling zone $\bar{\mathbf{Q}}_{shoal}$ is $O(E^2)$, and hence should be consistently neglected when compared with other wave-induced mean flow quantities. Hence in this preliminary study we shall ignore the effect of the mean sediment flux in $x > x_b$, and so $\bar{h} = h_0(x)$, $H = h_0(x) + \bar{\zeta}$. The steady wave set-up solution can now be derived in the usual manner from the mean momentum equation (11), see Mei (1983) or Svendsen (2006) for instance.

However, it transpires that now, in the surf zone U 6= 0, and in order to match at $x = x_b$, we may need to allow for U 6= 0 in the shoaling zone as well. Then, the conservation of mass equation implies that $HU = M = H_bU_b$, where the subscripts denotes the values at $x = x_b$. Then, for this steady onedimensional case, the mean momentum equation (11) yields

$$-\frac{M^2 H_x}{H^2} + S_x + gH\bar{\zeta}_x = 0, \quad (21)$$

where $S = \frac{c_g}{c}E\cos^2\phi + (\frac{c_g}{c} - \frac{1}{2})E . \quad (22)$

Here φ is the angle between the wave direction and the onshore direction, and S is the "xx" component of the tensor $\mathbf{S}$. In shallow water, $\varphi \approx 0, c_g \approx c, S \approx 3E/2$, and using (19) which expresses a, and hence E = in terms of h, we find that

$$\bar{\zeta} = -\frac{a^2}{4h} - \frac{M^2}{2gh^2} . \qquad (23)$$

Here we have assumed that $\bar{\zeta}$ is zero far offshore. When there is no sediment transport, $M = 0$ and this reduces to the well-known result obtained by Longuet-Higgins and Stewart (1962) of wave set-down in the shoaling zone Note that the extra term here always enhances the set-down. We show below that when $\nu << 1$, M is order ν, where $\nu << 1$ is defined by (27) below; hence this extra term can usually be neglected.

3.2 Surf zone formulation

In the surf zone $x_s < x < x_b$, we assume that $h_{0x} > 0$, and make the usual assumption (see Mei (1983) for instance) that the breaking wave height $2a$ is proportional to the total depth H, so that

$2a = \gamma H$, or $E = \frac{g\gamma^2 H^2}{8}$. (24)

Here the constant γ is determined empirically, and a typical value is γ = 0.88.. The radiation stress term S is again evaluated by $S = 3E/2$, so that from (24), $S = \Gamma g H^2/2$ where $\Gamma = 3\gamma^2/8$.
Next, we assume, as noted before, that the mean flow in the surf zone is dominated by the Stokes drift velocity $\mathbf{U}_s = (U_s,0)$. Hence in the expression (18) we assume that $U \approx U_s$. For the wave field in the surf zone we use the scaling $|\mathbf{u}_w|/(gH)^{1/2} \sim 2a/H$, so that using the empirical expression (24) ,

$|\mathbf{u}_w| \sim \gamma(gH)^{1/2}$. Similarly, we estimate that $U_S \sim -|\mathbf{u}_w|^2/\sqrt{gH}$, and so $U_s \sim -\gamma^2(gH)^{1/2}$. Again omitting the diffusive terms in (18), and setting $\bar{\mathbf{Q}} = (Q_{surf},0)$ we obtain the expression,

$$Q_{surf} = -\nu F(H), \quad (25)$$
$$F(H) = (1-\epsilon)H(gH)^{1/2}(1+\sigma(gH)^{3/2})$$
$$\text{where} \quad \nu = \frac{g\gamma^4 \mu \nu_b}{1-p_s}, \quad \sigma = \frac{\gamma \nu_s}{g\nu_b}.$$

, (26)
(27)

Using the estimates $\nu_b = 1.8 \times 10^{-4}\, s^2\, m^{-1}$, $\nu_s = 1.0 \times 10^{-3}\, s^3\, m^{-3}$ the dimensionless small parameter $\nu = 0.88 \times 10^{-4}$ and $\sigma = 0.05 s^3\, m^{-3}$.
The basic set of equations is then (16) using the expression (25), and (7 , 11), that is,

$$\bar{h}_t + \nu F(H)_x = 0, \quad (28)$$
$$H_t + (HU)_x = 0, \quad (29)$$
$$U_t + UU_x + g(1+\Gamma)H_x - g\bar{h}_x = 0. \quad (30)$$

Recall that $H = \bar{h} + \bar{\zeta}$.
First, we note that when $\nu = 0$, that is, the sediment transport term is removed, we see from (28) that $\bar{h} = h_0(x)$. A steady solution then exists with $U = 0$ from (29), and so also then $M = 0$. Equation (30 then reduces to

$\Gamma H H_x + H(H - h_0)_x = 0$, so that $H = H_b + \frac{h_0 - h_b}{(1+\Gamma)}$, (31)

where the constant $H_b = h_b + \bar{\zeta}_b$ is determined by requiring continuity of the total mean height at $x = x_b$. This is the well-known expression for wave set-up in the surf zone, see Mei (1983) or Svendson (2006). Note that the expression (31) is valid for any depth $h(x)$, although in the literature it is often derived only for a linear depth profile $h = \alpha x$. Also, using (23) with $M = 0$, $H_b = h_b - F_0/4h_b^{3/2}$, and since H_b must be positive, there is a restriction $F_0 < 4h_b^{5/2}$ on either the offshore wave amplitude a_0 or a_∞ through F_0, or on the breaker depth h_b, for this wave set-up solution to hold. The shoreline position $x = x_s$ can now be found by setting $H = 0$ in (31).
Next, since $U = 0$ when $\nu = 0$ (that is, there is no sediment transport), we anticipate that when $\nu << 1$, then U is $O(\nu)$, but importantly is not zero. It transpires that for the wave set-up solution of interest this is indeed the case. Before proceeding we return to the full expressions (18) and note that the criteria for the neglect of the diffusive terms are $\lambda_b\alpha << 1, \lambda_s\alpha < (H/g)^{1/2}$ where α is a measure of the beach slope $\bar{h}_x$. With $\lambda_b = 0.7$, $\lambda_s = 2.5s$, $H = 2m$ this implies that we require that $\alpha << 1.4, 0.2$ respectively.

3.3 Surf zone solution
The system (28, 29, 30) is a 3 × 3 nonlinear hyperbolic system of the form
$\mathbf{v}_t + \mathbf{A}(\mathbf{v})\mathbf{v}_x = 0$, $\mathbf{v}^t = [H,U,\bar{h}]$. (32)
The eigenvalues λ of $\mathbf{A}$ for this system are given by
$\det[\mathbf{A}(\mathbf{v}) - \lambda \mathbf{I}] = 0$, (33)
which leads to the cubic equation,
$\lambda\{g[1+\Gamma]H - (U-\lambda)^2\} - \nu g H F^0(H) = 0$ (34)
The system is hyperbolic if this expression has three real roots. For $\nu << 1$, the roots are

$$\lambda_1 \approx \frac{\nu g H F'(H)}{\{g(1+\Gamma)H - U^2\}} \quad \lambda_{2.3} \approx U \pm [g(1+\Gamma)H]^{1/2}. \quad (35)$$

All are real-valued, and so in this limit the system is hyperbolic. The first root is the one of main interest here, as it is arises directly from the sediment transport term.
Nonlinear hyperbolic systems support a family of simple wave solutions, of the form

$\mathbf{v} = \mathbf{v}(\alpha)$, (36)
where $\alpha = \alpha(x,t)$ is an arbitrary new variable, and could be taken as any one of the set $H,U,\bar{h}$. Substitution into (32) shows that
$\alpha_t + c(\alpha)\alpha_x = 0$, where $c = \lambda$, (37)
is one of the eigenvalues of **A**, and $\mathbf{v}_\alpha$ is then a corresponding eigenvector given by

$$-\lambda \bar{h}_\alpha + \nu F^0(H)H_\alpha = 0, \quad (38)$$

$$(U - \lambda)H_\alpha + HU_\alpha = 0, \quad (39)$$

$$(U - \lambda)U_\alpha + g(1 + \Gamma)H_\alpha - g\bar{h}_\alpha = 0. \quad (40)$$

We choose $\lambda = \lambda_1$, the root corresponding to the sediment transport term, and given approximately by (35) when $\nu << 1$. We then readily find that

$$c = \frac{\nu F'(H)}{1+\Gamma} + \cdots , \quad (41)$$

$$U = \frac{\nu F(H)}{(1+\Gamma)H} + \cdots$$

$$H = \frac{(\bar{h} + C_0)}{(1+\Gamma)} + \cdots .$$

, (42)
(43)
Here C_0 is a constant of integration, determined from a boundary matching condition. Because the relation (43) is conserved through shocks, see (51) below, this can be applied at $x = x_b$ so that,
$C_0 = \Gamma h_b + (1 + \Gamma)\bar{\zeta}_b$, (44)
so that $H = H_b + \dfrac{\bar{h} - h_b}{(1+\Gamma)} + \cdots,$ (45)
which is the same expression as (31) with h replaced by $\bar{h}$. Note that U is $O(\nu)$ with $U > 0$. Thus in this simple wave solution, the mean flow is weak and offshore.
A hyperbolic system can also support discontinuous, or shock, solutions. Assuming that across a discontinuity, the sediment flux, mass and momentum are conserved, the shock conditions can be derived in the usual way by integrating across the discontinuity. If the shock speed is V, then these are readily obtained from the set (28, 29, 30) ,

$$-V[\bar{h}] + \nu[F(H)] = 0 , \quad (46)$$

$$-V[H] + [HU] = 0,$$

$$-V[U] + [\frac{U^2}{2} + g(1+\Gamma)H - g\bar{h}] = 0.$$

(47)
(48)
Here $[\cdot]$ denotes the jump across the discontinuity. When $\nu << 1$, we see that the shock speed V of interest is $O(\nu)$. Assuming that, as above, also U is $O(\nu)$, the shock relations are approximated by

$$-V[\bar{h}] + \nu[F(H)] = 0, \quad (49)$$

$$-V[H] + [HU] = 0, \quad (50)$$

$$(1 + \Gamma)[H] - [\bar{h}] = 0. \quad (51)$$

Only equation (50) involves U, and equation (51) can be used to eliminate $\bar{h}$ so that (49, 50) reduce to

$$-V(1 + \Gamma)[H] + \nu[F(H)] = 0 \quad (52)$$

$$[(1 + \Gamma)HU - \nu F(H)] = 0. \quad (53)$$

Since $F(H)$ is an increasing function of H, the expression (52) shows that the shock speed V is positive. Also the expressions (53) shows that the simple wave relation (42) for U is conserved across the shock.
In general, the system (32) is to be solved with the boundary conditions at $x = x_b$ that $[H,U,\bar{h}] = [H_b,M/H_b,h_b]$, where $H_b = h_b + \bar{\zeta}_b$. When we assume that the solution in the shoaling zone $x > x_b$ is steady, given by (23) , then these boundary data are all known constants. The initial condition is more problematic to specify. Here we shall assume that in the surf zone $[H,U,\bar{h}] = [H_0(x),0,h_0(x)]$ at $t = 0$, where $H_0(x)$ is the solution (31) in the absence of

any sediment transport. This choice corresponds to turning on the sediment transport at $t = 0$, and is clearly an over-simplification of reality, but we expect the solution to be indicative of more realistic initial conditions. In effect, we are assuming that the wave field is turned on at $t = 0$ and reaches the sediment-free solution (31) instantaneously. That is we are assuming that the time scale for the steady sediment-free solution to be reached is much shorter than that for the sediment transport terms to take effect. Note that there is a discontinuity in U at the point $x = x_b, t = 0$, which requires that a shock emanates from this point. In the limit of interest when $\nu << 1$, the shock speed $V = O(\nu) > 0$, and we infer that a thin layer develops near $x = x_b$ but lying in $x > x_b$. In order to use the simple wave solution discussed above when $\nu << 1$ to solve this problem, we see that in this solution U 6= 0, albeit of $O(\nu)$. We infer that there is a thin boundary layer near $t = 0$ within which there an adjustment for U from zero to the simple wave value.

The simple wave solution can now be found by the method of characteristics, that is

$$\frac{dx}{dt} = c(\bar{h}), \quad \frac{d\bar{h}}{dt} = 0, \quad \text{for} \quad x < x_b, \quad (54)$$

$$\bar{h} = h_b \quad \text{at} \quad x = x_b, \quad \bar{h} = h(x) \text{ at} \quad t = 0, \quad (55)$$

so that $x - c(\bar{h})t = x_0, \bar{h} = h(x_0)$ for $x_0 < x_b$, (56)

where x_0 is the initial value of x along each characteristic. The solution can be written in the form

$\bar{h} = h(x - c(\bar{h})t)$, for $x - c(h_b)t < x_b$. (57)

Since $c(\bar{h})$ is an increasing function of h, it can be shown that in this simple wave solution, $h, \bar{H}$ decrease at each fixed x as t increases, and also decrease for each fixed t as x decreases, since we have assumed $h_x(x) > 0$. Thus, in this solution the beach is continually replenished. Because the characteristics go offshore, they intersect a shock $x = x_S(t)$) emanating from $x = x_b, t = 0$, where the jump conditions (52) is imposed. Thus we get that

$$-V(H_b - H(x_S, t)) + \nu(F(H_b) - F(H(x_S, t)) = 0\,, \quad V = \frac{dx_S}{dt}\,. \quad (58)$$

Here $H(x_S,t)$ is obtained from the simple wave solution, and because the shock is thin, we have approximated the values of $H(x \to x_S+,t)$ as that at $x = x_b$ for simplicity. The expression (58) is a differential equation that determines the shock. This completes the solution. Also the expression (53) for U can now be used to find the value of M, that is

$$M = \frac{\nu F(H_b)}{(1+\Gamma)}\,. \quad (59)$$

A typical timescale can be estimated as the time for a characteristic from $x = 0$ to reach $x = x_b$, that is $t_s = x_b/c(\bar{h} = 0)$, or

$$t_s = \frac{(1+\Gamma)x_b}{\nu F'(H_0(0))}\,, \quad H_0(0) = \frac{C_0}{1+\Gamma} = H_b - \frac{h_b}{(1+\Gamma)}\,. \quad (60)$$

For instance, with the parameter values already specified, and with $x_b = 20m, H_b = 2m, h_b = 1.8m$, we find that $t_s = 4800s$ which is a reasonable value. THis estimate is dominated by the suspended sediment term, that with coefficient σ. If this is removed the estimated timescale is 7 times larger.

For a linear beach slope $h_0 = \alpha x$, the simple wave solution (57) reduces to

$$\bar{h} = \alpha(x - c(\bar{h})t), \quad (61)$$

For $\nu << 1$ this reduces to the expression

$$(1+\Gamma)H = \alpha x + C_0 - \frac{2\nu\alpha F'(H)t}{1+\Gamma}, \quad (62)$$

where $F'(H) = (1-\epsilon)\{\frac{3(gH)^{1/2}}{2} + 3\sigma(gH)^2\}$.

The first expression reduces to (31) when $\nu \to 0$, but for $\nu > 0$ it is a quartic polynomial equation for $Y = (gH)^{1/2}$, which can be written as

$$Y^2 = X - 2T(Y + 2\gamma\sigma Y^4)\,, \quad T = \frac{3\nu\alpha g t}{2(1+\Gamma)^2}\,, \quad X = \frac{g(\alpha x + C_0)}{1+\Gamma}\,. \quad (63)$$

Explicit solutions are not readily available, but it is readily shown that there is only one positive root for Y , for each given $X > 0, T > 0$. Also, the shoreline, where $Y = 0$ does note change in time, and remains at $X = 0$. However, explicit solutions can be found in the two limits when either bedload or suspended sediment dominate. These correspond formally to the respective limits $\sigma \to 0, \sigma \to \infty$, when we get that

$$Y = \frac{X}{T + (T^2 + X)^{1/2}}\,, \quad \text{or } Y^2 = \frac{2X}{1 + (1 + 16\gamma\sigma XT)^{1/2}}\,, \quad (64)$$

respectively. Note that the limit $\sigma \to \infty$ is non-dimensional form is $\sigma_{nd} = \sigma(gH_b)^{3/2} \to \infty$, and is only valid when $Y,X > 0$. These are plotted in figure 1 for $\bar{h}$ as a function of h for a fixed value of $t = 10^4 s$, where for the limit $\sigma \to \infty$, the non-dimensional $\sigma_{nd} = 43.5$ for the plot in this figure.

3.4 Steady state

It is clear from (28, 29) that the present sediment transport model cannot allow any steady state to form, as $\bar{h}_t = H_t = 0$ would then imply that $H = 0$, which is unacceptable. Hence, if a steady state is to be reached, we must replace the sediment law (25) by an expression which takes account of the beach slope through the diffusive terms in (18). Thus, from the discussion in section 3.2 we now replace (25) with

$$Q_{surf} \approx \nu\{-F(H) + D(H)b_x\},$$

$$D(H) = \frac{\lambda_b}{\gamma} H(gH)^{1/2} + \sigma\lambda_s\gamma^2 gH^2(gH)^{1/2} \quad (65)$$

.

Consequently equation (28) is replaced by

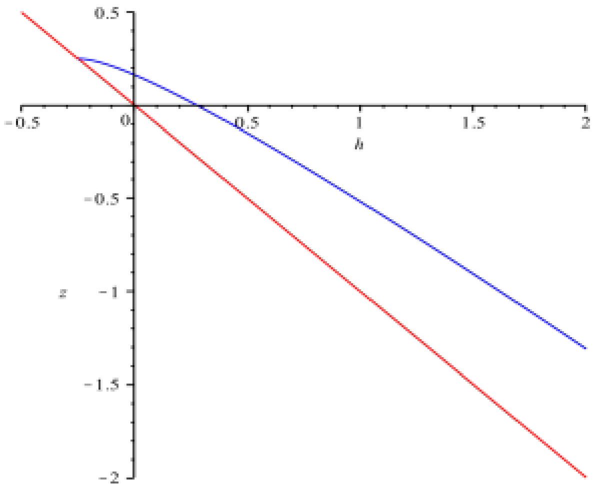
$\bar{h}_t = \nu\{-F(H) + D(H)b_x\}_x$, $b = \bar{h} - h_r(x)$. (66)

The remaining two equations (29, 30) are unchanged. Equation (66) has the structure of a nonlinear diffusion equation, and so there is a possibility that a steady-state can be achieved. Indeed if we assume that there is a steady-state solution then $U = 0$, (66) implies that $\bar{Q}_{surf} = 0$, and then

$$F(H) - D(H)b_x = 0. \quad (67)$$

Equation (30) can be integrated to yield

$$H(1 + \Gamma) = \bar{h} + \text{constant}. \quad (68)$$



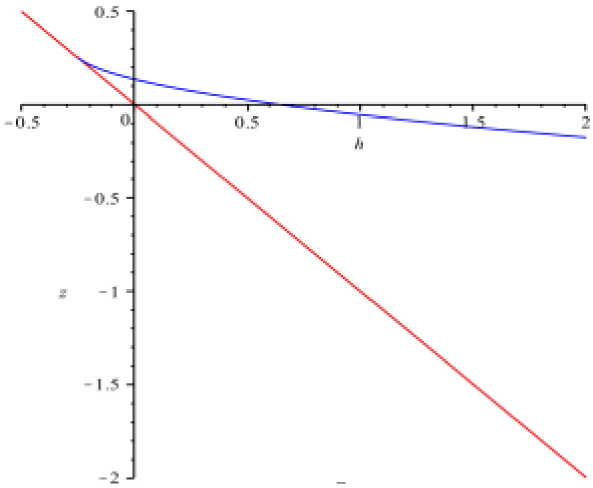


Figure 1: Plot of the solution (64) for $\bar{h}$ (blue) for the parameter setting

$\gamma = 0.88, \alpha = 0.1, h_b = 2\,m, H_b = 1.73\,m, \epsilon = 0$ for $t = 10^4\,s$ for the case $\sigma = 0$ in the upper panel, and for the case $\sigma = 0.5 s^3 m^{-3}$ in the lower panel.
In each case the red line is h, the value at $t = 0$.
Substituting (68) into (67) yields
$1 + \gamma\sigma(gH)^{3/2} = (\lambda_b + \sigma\lambda_s\gamma gH)((1 + \Gamma)H_x - h_{rx})$. (69)

Clearly the solution will depend on the choice of the reference depth, and here we make the simple choice that $h_r(x)$ = constant + $\alpha_r x$. The general solution can now be found by quadrature. However it is more instructive to consider the two limits $\sigma \to 0,\infty$, which correspond to the cases when either bed-load or suspended transport dominates. These limits yield respectively

$$\sigma = 0: \quad H = \frac{\tilde{\alpha}(x - x_s)}{(1+\Gamma)}, \quad \tilde{\alpha} = \alpha_r + \frac{\gamma}{\lambda_b}, \qquad (70)$$

$$\sigma \to \infty: \ (gH)^{1/2} - D_0 \log(1 + \frac{(gH)^{1/2}}{D_0}) = \frac{\gamma(x - x_s)}{\lambda_s(1+\Gamma)}, \qquad (71)$$

where $D_0 = \dfrac{\alpha_0 g \lambda_s}{\gamma}$.

Here $x = x_s$ is the shoreline, and the corresponding expressions for $\bar{h}$ are recovered from (68). In the case $\sigma = 0$, the profile is just a linear slope but enhanced over the reference slope α_r. In the case $\sigma \to \infty$, we note that when $\alpha_r = 0, D_0 = 0$ and (70) reduces to a quadratic expression in $H = C_1(x - x_s)^2, C_1 = \gamma^2/g\lambda_s^2(1+\Gamma)^2$, while when $D_0 \to \infty$, the profile is again a quadratic expression, but now $4H \to C_1(x - x_s)^2$. Also, for this same case as $H \to 0$, again $4H \to C_1(x - x_s)^2$, while as $H \to \infty$, $H \to C_1(x - x_s)^2$. In effect the entire solution is close to some parabolic profile. For intermediate values of σ the solution varies between the linear slope (70) and the expression defined by (71). We infer that these equilibrium beach profiles range between a linear and a parabolic profile, and can probably be well approximated by a power law $(x - x_s)^\beta, 1 \leq \beta \leq 2$. However, it seems that the well-known Dean's law (Dean 1991, Dean and Darlymple 2002) when the profile is proportional to $(x - x_s)^{2/3}$ is not described by the present class of solutions. Indeed, to obtain Dean's law by the present approach requires that $D(H)/F(H) \propto (gH)^{1/2}$, Examining the formula (18), we see that this would require a stronger dependence on $|\mathbf{u}_w|$ in the diffusive term than this formula allows for. Whether or not the unsteady simple wave solutions of the previous subsections will eventually reach a steady state requires numerical solutions of (29, 30, 66), and will not be investigated here.

IV. SUMMARY AND DISCUSSION

In this paper we have augmented the usual wave-averaged mean field equations, described in section 2, commonly used to describe wave set-up and wave-induced mean currents in the near-shore zone, with a sediment flux law (18), which has a form similar to several available in the literature. In this present model, any sediment movement in the shoaling zone is ignored as being $O(E^2)$, and instead our focus is on how the augmented model modifies wave set-up in the surf zone. Here the sediment flux law is modelled empirically, based on (18), but with a modification to reflect the dominant effect of the Stokes drift term, leading to (25). Our main result in section 3.3 is that, when the diffusive terms in the flux law are ignored, then there is no steady-state set-up, and instead the mean bottom depth $\bar{h}$ in the surf zone evolves according to a simple wave equation. This is solved to yield a prediction that the beach is replenished. In section 3.4 we show that if the diffusive terms in the sediment flux law (25) are retained, then the simple wave equation, whose solutions are intrinsically unsteady, is replaced by a nonlinear diffusion equation (66) which allows a steady-state solution. This can be well represented by a power-law profile with index varying between one and two, that is between linear and parabolic profiles.

Although our present model makes a specific choice of the empirical parameters in (18), we would expect that other choices will lead to qualitatively similar results to those obtained here. A more serious limitation of the present model is that the outer boundary of the surf zone $x = x_b$ is assumed here to be fixed for all time. When sediment transport is allowed, the wave set-up becomes unsteady, and our solution indeed indicates that x_b will also be unsteady, and migrate offshore as the mean total depth decreases in the surf zone. This issue will await future study. Also, the present model is entirely one-dimensional, and it would be interesting to examine the stability of the solutions found here to transverse perturbations,

REFERENCES

[1] Blondeaux, P., (2001), Mechanics of Coastal Forms, *Annual Review of Fluid Mechanics*, vol.33, pg 339-370.

[2] Bowen, A.J., Inman, D.L. and Simmons, V.P. (1968), Wave set-down and set-up, *J. Geophys. Res*, **73**, 2569-2577.

[3] Caballeria, M., Coco, G., Falques, A. and Huntley, D. A. (2002), Selforganization mechanisms for the formation of nearshore crescentic and transverse sand bars. *J. Fluid Mech., 465*, 379- 410.

[4] Calvete, D., Falques, A., De Swart, H. E. and Walgreen, M. (2001) , Modelling the formation of shoreface-connected sand ridges on stormdominated inner shelves, *J. Fluid Mech.*, **441**, 169-193.

[5] Calvete, D., De Swart, H. E. and Falques, A. (2002), Effect of depthdependent wave stirring on the final amplitude of shoreface-connected sand ridges, *Cont. Shelf Res.*, **22**, 2763-2776.

[6] Coastal Engineering Manual (2002). *Department of the Army. U.S. Army Corps of Engineers, Manual No. 1110-2-1100*, **Ch. 3**.

[7] Dean, R. G. (1991), Equilibrium beach profiles: Characteristics and applications, *J. Coastal Research*, **7**, 53-84.

[8] Dean, R. G. and Dalrymple, R. A. (2002), *Coastal Processes with Engineering Applications*, CUP, Cambridge.

[9] Garnier, R., Calvete, D., Falques, A. and Caballeria, M. (2006) Generation and nonlinear evolution of shore-oblique/transverse sand bars. *J. Fluid Mech.*, **567**, 327-360.

[10] Garnier, R., Calvete, D., Falque, A. and Dodd, N. (2008), Modelling the formation and the long-term behavior of rip channel systems from the deformation of a longshore bar, *J. Geophy. Res.*, **113**, C07053.

[11] Hancock, M. J., Landry B.J. and Mei, C.C. (2008), Sandbar formation under surface waves: Theory and experiments, *J. Geophys. Res.*, **113**, C07022.

[12] Lane, E.M and Restrepo, J.M (2007), Shoreface-connected ridges under the action of waves and currents, *J. of Fluid Mechanics*, **582**, pp. 23-52.

[13] Longuet-Higgins, M.S and Stewart, R.W. (1962), Radiation stress and mass transport in gravity waves, with application to surf beats, *J. Fluid Mech.*, **13**, 481-504.

[14] McCall, R.T., van Thiel de Vries, J. S.M., Plant, N.G., van Dongeren, A.R., Roelvink, J. A., Thompson, D. M. and Reniers, A.J.H.M. (2010) , Two-dimensional time dependent hurricane overwash and erosion modeling at Santa Rosa Island, *Coastal Engineering*, **57**, 668-683.

[15] Mei, C.C. (1983), *Applied Dynamics of Ocean Surface Waves*, Wiley, New York.

[16] Restropo, J.M. and Bona, J.L. (1995) Three-dimensional model for the formation of longshore sand structures on the continental shelf, *Nonlinearity*, **8**, 781-820.

[17] Restrepo, J. M. (2001), Wave-current interactions in shallow waters and shore-connected ridges, *Continental Shelf Research*, **21**, 1331-1360.

[18] Roelvink, D., Reniers, A., van Dongeren, A., van Thiel de Vries, J., McCall, R. and Lescinski, J. (2009) Modelling storm impacts on beaches, dunes and barrier islands, *Coastal Engineering*, **56**, 1133-1152.

[19] Roelvink, J.A. and Broker, I. (1993), Cross-shore profile models, *Coastal Engineering*, **21**, 163-191.

[20] Svendsen, Ib A. (2006) *Introduction to nearshore hydrodynamics*, World Scientific, Singapore.

[21] Walgreen, M., Calvete, D. and de Swart, H.E. (2002), Growth of largescale bed forms due to storm-driven and tidal currents: a model approach, *Cont. Shelf Res.*, **22**, 2777-2793.