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Location of Regions Containing all or Some Zeros of a Polynomial

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ABSTRACT
In this paper we locate the regions which contain all or some of the zeros of a polynomial when the coefficients of the polynomial are restricted to certain conditions.
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I. INTRODUCTION

The following result known as the Enestrom-Kakeya Theorem [3] is of fundamental importance on locating a region containing all the zeros of a polynomial with monotonically increasing positive coefficients:

Theorem A: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that

$$a_n \ge a_{n-1} \ge ...... \ge a_1 \ge a_0 > 0 .$$

Then all the zeros of P(z) lie in $|z| \le 1$.

In the literature several generalizations and extensions of this result are available [].
Recently Gulzar et al [1,2] proved the following results:

Theorem B: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with

$\operatorname{Re}(a_j)=\alpha_j, \operatorname{Im}(a_j)=\beta_j, j=0,1,2,......,n$ such that for some $\lambda, 0 \le \lambda \le n-1$ and for some $k \ge 1, o < \tau \le 1$,

$$k\alpha_n \ge \alpha_{n-1} \ge ...... \ge \tau\alpha_\lambda$$

and

$$L=|\alpha_\lambda-\alpha_{\lambda-1}|+|\alpha_{\lambda-1}-\alpha_{\lambda-2}|+......+|\alpha_1-\alpha_0|+|\alpha_0| .$$

Then all the zeros of P(z) lie in

$$\left|z+\frac{(k-1)\alpha_n}{a_n}\right| \le \frac{k\alpha_n-\tau\alpha_\lambda+(1-\tau)|\alpha_\lambda|+L+2\sum_{j=0}^{n}|\beta_j|}{|a_n|} .$$

Theorem C: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with

$\operatorname{Re}(a_j)=\alpha_j, \operatorname{Im}(a_j)=\beta_j, j=0,1,2,......,n$ such that for some $\lambda, 0 \le \lambda \le n-1$ and for some $k \ge 1, o < \tau \le 1$,

$$k\alpha_n \ge \alpha_{n-1} \ge ...... \ge \alpha_{\lambda+1} \ge \tau\alpha_\lambda$$

and

$$L=|\alpha_\lambda-\alpha_{\lambda-1}|+|\alpha_{\lambda-1}-\alpha_{\lambda-2}|+......+|\alpha_1-\alpha_0|+|\alpha_0| .$$

Then the number f zeros of P(z) in $|z| \le \delta, 0 < \delta < 1$ does not exceed

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{|a_n|+(k-1)|\alpha_n|+k\alpha_n-\alpha_\lambda+L+(1-\tau)|\alpha_\lambda|+2\sum_{j=0}^{n}|\beta_j|}{|a_0|}.$$

II. MAIN RESULTS

In this paper we prove the following generalizations of Theorems B and C:

Theorem 1: Let $P(z)=\sum_{j=0}^{n}a_jz^j$ be a polynomial of degree n with

$\mathrm{Re}(a_j)=\alpha_j$, $\mathrm{Im}(a_j)=\beta_j$, $j=0,1,2,......,n$ such that for some $\lambda, 0\le\lambda\le n-1$ and for some $k_1,k_2\ge 1, o<\tau\le 1$,

$$k_1\alpha_n\ge k_2\alpha_{n-1}\ge ......\ge\tau\alpha_\lambda$$

and

$$L=|\alpha_\lambda-\alpha_{\lambda-1}|+|\alpha_{\lambda-1}-\alpha_{\lambda-2}|+......+|\alpha_1-\alpha_0|+|\alpha_0|.$$

Then all the zeros of P(z) lie in

$$\left|z+\frac{(k_1-1)\alpha_n}{a_n}-\frac{(k_2-1)\alpha_{-1}}{a_n}\right|\le\frac{k_1\alpha_n+(k_2-1)|\alpha_{n-1}|+|\alpha_\lambda|+L-\tau(|\alpha_\lambda|+\alpha_\lambda)+2\sum_{j=0}^{n}|\beta_j|}{|a_n|}.$$

Theorem 2: Let $P(z)=\sum_{j=0}^{n}a_jz^j$ be a polynomial of degree n with

$\mathrm{Re}(a_j)=\alpha_j$, $\mathrm{Im}(a_j)=\beta_j$, $j=0,1,2,......,n$ such that for some $\lambda, 0\le\lambda\le n-1$ and for some $k_1,k_2\ge 1, o<\tau\le 1$,

$$k_1\alpha_n\ge k_2\alpha_{n-1}\ge ......\ge\alpha_{\lambda+1}\ge\tau\alpha_\lambda$$

and

$$L=|\alpha_\lambda-\alpha_{\lambda-1}|+|\alpha_{\lambda-1}-\alpha_{\lambda-2}|+......+|\alpha_1-\alpha_0|+|\alpha_0|.$$

Then the number of zeros of P(z) in $|z|\le\delta, 0<\delta<1$ does not exceed

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{1}{|a_0|}[|a_n|+(k_1-1)|\alpha_n|+k_1\alpha_n+2(k_2-1)|\alpha_{n-1}|+|\alpha_\lambda|+L-\tau(\alpha_\lambda+|\alpha_\lambda|)+2\sum_{j=0}^{n}|\beta_j|.$$

Remark 1: Taking $k_2=1$, Theorem 1 reduces to Theorem B and Theorem 2 reduces to Theorem C.
For different values of the parameters in Theorems 1 and 2 , we get many interesting results. For example taking $\tau=1$, Theorem 1 gives the following result:

Corollary 1: Let $P(z)=\sum_{j=0}^{n}a_jz^j$ be a polynomial of degree n with

$\mathrm{Re}(a_j)=\alpha_j$, $\mathrm{Im}(a_j)=\beta_j$, $j=0,1,2,......,n$ such that for some $\lambda, 0\le\lambda\le n-1$ and for some $k_1,k_2\ge 1,$,

$$k_1\alpha_n\ge k_2\alpha_{n-1}\ge ......\ge\alpha_\lambda$$

and

$$L=|\alpha_\lambda-\alpha_{\lambda-1}|+|\alpha_{\lambda-1}-\alpha_{\lambda-2}|+......+|\alpha_1-\alpha_0|+|\alpha_0|.$$

Then all the zeros of P(z) lie in

$$\left| z + \frac{(k_1 - 1)\alpha_n}{a_n} \right| \le \frac{k_1\alpha_n + (k_2 - 1)(\alpha_{n-1} + |\alpha_{n-1}|) + L - \alpha_\lambda + 2\sum_{j=0}^{n} |\beta_j|}{|a_n|} .$$

Taking $\tau = 1$, Theorem 2 gives the following result:

Corollary 2: Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $\mathrm{Re}(a_j) = \alpha_j, \mathrm{Im}(a_j) = \beta_j, j = 0,1,2,......, n$ such that for some $\lambda, 0 \le \lambda \le n-1$ and for some $k_1, k_2 \ge 1$, ,

$$k_1\alpha_n \ge k_2\alpha_{n-1} \ge ...... \ge \alpha_{\lambda+1} \ge \alpha_\lambda$$

and

$$L = |\alpha_\lambda - \alpha_{\lambda-1}| + |\alpha_{\lambda-1} - \alpha_{\lambda-2}| + ... + |\alpha_1 - \alpha_0| + |\alpha_0| .$$

Then the number f zeros of P(z) in $|z| \le \delta, 0 < \delta < 1$ does not exceed

$$\frac{1}{\log \frac{1}{\delta}} \log \frac{1}{|a_0|}[|a_n| + (k_1 - 1)|\alpha_n| + k_1\alpha_n + 2(k_2 - 1)|\alpha_{n-1}| + L - \alpha_\lambda + 2\sum_{j=0}^{n} |\beta_j|].$$

III. LEMMA

For the proof of Theorem 2, we need the following result:

Lemma: Let f (z) be analytic for $|z| \le 1$, $f(0) \ne 0$ and $|f(z)| \le M$ for $|z| \le 1$. Then the number of zeros of f(z) in $|z| \le \delta, 0 < \delta < 1$ does not exceed $\frac{1}{\log \frac{1}{\delta}} \log \frac{M}{|f(0)|}$.

(for reference see [4]).

3. Proofs of Theorems

Proof of Theorem 1: Consider the polynomial

$$\begin{aligned}
F(z) &= (1 - z)P(z) \\
&= (1 - z)(a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0) \\
&= -a_n z^{n+1} + (a_n - a_{n-1})z^n + ... + (a_{\lambda+1} - a_\lambda)z^{\lambda+1} + (a_\lambda - a_{\lambda-1})z^\lambda \\
&\quad + ... + (a_1 - a_0)z + a_0 \\
&= -a_n z^{n+1} - (k_1 - 1)\alpha_n z^n + (k_1\alpha_n - k_2\alpha_{n-1})z^n + (k_2 - 1)\alpha_{n-1} z^n + (k_2\alpha_{n-1} - \alpha_{n-2})z^{n-1} \\
&\quad - (k_2 - 1)\alpha_{n-1} z^{n-1} + (\alpha_{n-2} - \alpha_{n-3})z^{n-2} + ... + (\alpha_{\lambda+1} - \tau\alpha_\lambda)z^{\lambda+1} + (\tau - 1)\alpha_\lambda z^{\lambda+1} \\
&\quad + (\alpha_\lambda - \alpha_{\lambda-1})z^\lambda + ... + (\alpha_1 - \alpha_0)z + \alpha_0 + i\{(\beta_n - \beta_{n-1})z^n + ... \\
&\quad + (\beta_1 - \beta_0)z + \beta_0\}
\end{aligned}$$

For $|z| > 1$ so that $\frac{1}{|z|^j} < 1, \forall j = 1,2,......, n$, we have, by using the hypothesis

$$\begin{aligned}
|F(z)| \ge{} & |a_n z + (k_1 - 1)\alpha_n - (k_2 - 1)\alpha_{n-1}||z|^n - [|k_1\alpha_n - k_2\alpha_{n-1}||z|^n + |k_2 - 1||\alpha_{n-1}||z|^{n-1} + |k_2\alpha_{n-1} - \alpha_{n-2}|.|z|^{n-1} \\
& + (\alpha_{n-2} - \alpha_{n-3})|z.|^{n-2} + ... + |\alpha_{\lambda+1} - \tau\alpha_\lambda||z|^{\lambda+1} + |\tau - 1||\alpha_\lambda||z|^{\lambda+1} + |\alpha_\lambda - \alpha_{\lambda-1}||z|^\lambda \\
& + ... + |\alpha_1 - \alpha_0||z| + |\alpha_0| + \{|\beta_n - \beta_{n-1}||z|^n + ... + |\beta_1 - \beta_0||z| + |\beta_0|]
\end{aligned}$$

$$\begin{aligned}
&= |z|^{n}[|a_{n}z + (k_{1}-1)\alpha_{n} - (k_{2}-1)\alpha_{n-1}| - \{|k_{1}\alpha_{n} - k_{2}\alpha_{n-1}| + \frac{(k_{2}-1)|\alpha_{n-1}|}{|z|} + \frac{|k_{2}\alpha_{n-1} - \alpha_{n-2}|}{|z|} \\
&\quad + \frac{|\alpha_{n-2} - \alpha_{n-3}|}{|z|^{2}} + \ldots\ldots + \frac{|\alpha_{\lambda+1} - \tau\alpha_{\lambda}|}{|z|^{n-\lambda-1}} + \frac{(1-\tau)|\alpha_{\lambda}|}{|z|^{n-\lambda-1}} + \frac{|\alpha_{\lambda} - \alpha_{\lambda-1}|}{|z|^{n-\lambda}} \\
&\quad + \ldots\ldots + \frac{|\alpha_{1} - \alpha_{0}|}{|z|^{n-1}} + \frac{|\alpha_{0}|}{|z|^{n}} + |\beta_{n} - \beta_{n-1}| + \frac{|\beta_{n-1} - \beta_{n-2}|}{|z|} + \ldots\ldots \\
&\quad + \frac{|\beta_{1} - \beta_{0}|}{|z|^{n-1}} + \frac{|\beta_{0}|}{|z|^{n}}\}] \\
&> |z|^{n}[|a_{n}z + (k_{1}-1)\alpha_{n} - (k_{2}-1)|\alpha_{n-1}|| - \{|k_{1}\alpha_{n} - k_{2}\alpha_{n-1}| + (k_{2}-1)|\alpha_{n-1}| \\
&\quad + |k_{2}\alpha_{n-1} - \alpha_{n-2}| + |\alpha_{n-2} - \alpha_{n-3}| + \ldots.. + |\alpha_{\lambda+1} - \tau\alpha_{\lambda}| + (1-\tau)|\alpha_{\lambda}| \\
&\quad + |\alpha_{\lambda} - \alpha_{\lambda-1}| + \ldots\ldots + |\alpha_{1} - \alpha_{0}| + |\alpha_{0}| + |\beta_{n} - \beta_{n-1}| + |\beta_{n-1} - \beta_{n-2}| + \ldots\ldots \\
&\quad + |\beta_{1} - \beta_{0}| + |\beta_{0}|\}] \\
&\geq |z|^{n}[|a_{n}z + (k_{1}-1)\alpha_{n} - (k_{2}-1)|\alpha_{n-1}|| - \{k_{1}\alpha_{n} - k_{2}\alpha_{n-1} + (k_{2}-1)|\alpha_{n-1}| \\
&\quad + k_{2}\alpha_{n-1} - \alpha_{n-2} + \alpha_{n-2} - \alpha_{n-3} + \ldots.. + \alpha_{\lambda+1} - \tau\alpha_{\lambda} + (1-\tau)|\alpha_{\lambda}| \\
&\quad + |\alpha_{\lambda} - \alpha_{\lambda-1}| + \ldots\ldots + |\alpha_{1} - \alpha_{0}| + |\alpha_{0}| + |\beta_{n}| + |\beta_{n-1}| + |\beta_{n-1}| + |\beta_{n-2}| + \ldots\ldots \\
&\quad + |\beta_{1}| + |\beta_{0}| + |\beta_{0}|\}] \\
&\geq |z|^{n}[|a_{n}z + (k_{1}-1)\alpha_{n} - (k_{2}-1)\alpha_{n-1}| - \{k_{1}\alpha_{n} + (k_{2}-1)|\alpha_{n-1}| + |\alpha_{\lambda}| + L \\
&\quad - \tau(|\alpha_{\lambda}| + \alpha_{\lambda}) + 2\sum_{j=0}^{n}|\beta_{j}|\}] \\
&> 0
\end{aligned}$$

if

$$|a_{n}z + (k_{1}-1)\alpha_{n} - (k_{2}-1)\alpha_{n-1}| > k_{1}\alpha_{n} + (k_{2}-1)|\alpha_{n-1}| + |\alpha_{\lambda}| + L - \tau(|\alpha_{\lambda}| + \alpha_{\lambda}) + 2\sum_{j=0}^{n}|\beta_{j}|$$

i.e. if

$$\left|z + \frac{(k_{1}-1)\alpha_{n}}{a_{n}} - \frac{(k_{2}-1)\alpha_{-1}}{a_{n}}\right| > \frac{k_{1}\alpha_{n} + (k_{2}-1)|\alpha_{n-1}| + |\alpha_{\lambda}| + L - \tau(|\alpha_{\lambda}| + \alpha_{\lambda}) + 2\sum_{j=0}^{n}|\beta_{j}|}{|a_{n}|}.$$

This shows that those zeros of F(z) whose modulus is greater than 1 lie in

$$\left|z + \frac{(k_{1}-1)\alpha_{n}}{a_{n}} - \frac{(k_{2}-1)\alpha_{-1}}{a_{n}}\right| \leq \frac{k_{1}\alpha_{n} + (k_{2}-1)|\alpha_{n-1}| + |\alpha_{\lambda}| + L - \tau(|\alpha_{\lambda}| + \alpha_{\lambda}) + 2\sum_{j=0}^{n}|\beta_{j}|}{|a_{n}|}.$$

Since the zeros of F(z) whose modulus is less than or equal to 1 already satisfy the above inequality and since the zeros of P(z) are also the zeros of F(z) , it follows that all the zeros of P(z) lie in

$$\left|z + \frac{(k_{1}-1)\alpha_{n}}{a_{n}} - \frac{(k_{2}-1)\alpha_{-1}}{a_{n}}\right| \leq \frac{k_{1}\alpha_{n} + (k_{2}-1)|\alpha_{n-1}| + |\alpha_{\lambda}| + L - \tau(|\alpha_{\lambda}| + \alpha_{\lambda}) + 2\sum_{j=0}^{n}|\beta_{j}|}{|a_{n}|}.$$

That proves Theorem 1.

Proof of Theorem 2: Consider the polynomial

$$\begin{aligned}F(z) &= (1-z)P(z)\\ &= (1-z)(a_n z^n + a_{n-1}z^{n-1} + \ldots\ldots + a_1 z + a_0)\\ &= -a_n z^{n+1} + (a_n - a_{n-1})z^n + \ldots\ldots + (a_{\lambda+1} - a_\lambda)z^{\lambda+1} + (a_\lambda - a_{\lambda-1})z^\lambda\\ &\quad + \ldots\ldots + (a_1 - a_0)z + a_0\\ &= -a_n z^{n+1} - (k_1-1)\alpha_n z^n + (k_1\alpha_n - k_2\alpha_{n-1})z^n + (k_2-1)\alpha_{n-1}z^n + (k_2\alpha_{n-1} - \alpha_{n-2})z^{n-1}\\ &\quad - (k_2-1)\alpha_{n-1}z^{n-1} + (\alpha_{n-2} - \alpha_{n-3})z^{n-2} + \ldots\ldots + (\alpha_{\lambda+1} - \tau\alpha_\lambda)z^{\lambda+1} + (\tau-1)\alpha_\lambda z^{\lambda+1}\\ &\quad + (\alpha_\lambda - \alpha_{\lambda-1})z^\lambda + \ldots\ldots + (\alpha_1 - \alpha_0)z + \alpha_0 + i\{(\beta_n - \beta_{n-1})z^n + \ldots\ldots\\ &\quad + (\beta_1 - \beta_0)z + \beta_0\}\end{aligned}$$

For $|z| \le 1$, we have, by using the hypothesis

$$\begin{aligned}|F(z)| &\le |a_n| + (k_1-1)|\alpha_n| + |k_1\alpha_n - k_2\alpha_{n-1}| + (k_2-1)|\alpha_{n-1}| + |k_2\alpha_{n-1} - \alpha_{n-2}| + (k_2-1)|\alpha_{n-1}|\\ &\quad + |\alpha_{n-2} - \alpha_{n-3}| + \ldots\ldots + |\alpha_{\lambda+1} - \tau\alpha_\lambda| + (1-\tau)|\alpha_\lambda| + |\alpha_\lambda - \alpha_{\lambda-1}| + \ldots\ldots + |\alpha_1 - \alpha_0| + |\alpha_0|\\ &\quad + |\beta_n - \beta_{n-1}| + |\beta_{n-1} - \beta_{n-2}| + \ldots\ldots + |\beta_1 - \beta_0| + |\beta_0|\\ &\le |a_n| + (k_1-1)|\alpha_n| + k_1\alpha_n - k_2\alpha_{n-1} + (k_2-1)|\alpha_{n-1}| + k_2\alpha_{n-1} - \alpha_{n-2} + (k_2-1)|\alpha_{n-1}|\end{aligned}$$

$$\begin{aligned}&+ \alpha_{n-2} - \alpha_{n-3} + \ldots\ldots + \alpha_{\lambda+1} - \tau\alpha_\lambda + (1-\tau)|\alpha_\lambda| + |\alpha_\lambda - \alpha_{\lambda-1}| + \ldots\ldots + |\alpha_1 - \alpha_0| + |\alpha_0| + |\beta_n|\\ &+ |\beta_{n-1}| + |\beta_{n-1}| + |\beta_{n-2}| + \ldots\ldots + |\beta_1| + |\beta_0| + |\beta_0|\\ &\le |a_n| + (k_1-1)|\alpha_n| + k_1\alpha_n + 2(k_2-1)|\alpha_{n-1}|) + |\alpha_\lambda| + L - \tau(\alpha_\lambda + |\alpha_\lambda|)\\ &\quad + 2\sum_{j=0}^{n}|\beta_j|\end{aligned}$$

Since F(z) is analytic for $|z| \le 1$, $F(0) = a_0 \ne 0$, it follows by the Lemma that the number of zeros of F(z) in $|z| \le \delta, 0 < \delta < 1$ des not exceed

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{1}{|a_0|}[|a_n| + (k_1-1)|\alpha_n| + k_1\alpha_n + 2(k_2-1)|\alpha_{n-1}| + |\alpha_\lambda| + L - \tau(\alpha_\lambda + |\alpha_\lambda|) + 2\sum_{j=0}^{n}|\beta_j|.$$

Since the zeros of P(z) are also the zeros of F(z) , it follows that the number of zeros of P(z) in $|z| \le \delta, 0 < \delta < 1$ des not exceed

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{1}{|a_0|}[|a_n| + (k_1-1)|\alpha_n| + k_1\alpha_n + 2(k_2-1)|\alpha_{n-1}|) + |\alpha_\lambda| + L - \tau(\alpha_\lambda + |\alpha_\lambda|) + 2\sum_{j=0}^{n}|\beta_j|.$$

That completes the proof of Theorem 2.

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