Generating Functions of Certain Hypergeometric functions By Means of Fractional Calculus

Manoj Singh¹, Sarita Pundhir², Mukesh PalSingh³

¹Department of Mathematics, Faculty of Science, Jazan University, Jazan, Saudi Arabia.
² Department of Mathematics, IMT Engineering College, Greater Noida, India.
³Department of Mathematics, Indraprastha Institute of Technology, Gajraula, J.P. Nagar, India.

Corresponding Author: Manoj Singh
E mail: manojsingh221181@gmail.com, msingh@jazanu.edu.sa

ABSTRACT

In the present investigation, we apply the fractional derivative techniques on some well-known identities to obtain linear generating functions for several classes of hypergeometric functions. Some special cases of results were also discussed in the end.

Mathematics Subject Classification (2010): Primary 42C05, Secondary 33C45.

Keywords: Fractional derivative, Generating functions, Appell function, Generalized Appell function, Horn type hypergeometric function, Lauricella function, Triplehypergeometric series, etc.

I. INTRODUCTION

Fractional calculus is concerned with the theory of derivatives and integrals of non-integer order. There are many applications of fractional derivatives in the theory of hypergeometric functions, in solving ordinary and partial differential equations and integral equations (see [4], [5], [7], [12]). Nowadays this subject have wide application in several scientific areas like control theory, physics and engineering, stochastic process, modeling, probability theory etc.

In 1731, Euler extended the derivative formula ([12] pp.285), to the general form as

$$D_z^\mu f(z) = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-n+1)} z^{\lambda-\mu}$$

where $\mu$ is an ordinary complex number.

We recall here the application of Euler derivative formula to some special functions. For this we use the theorem. 

**Theorem 1**: If a function $f(z)$ is analytic in the disc $|z| < \rho$, has the power series expansion,

$$f(z) = \sum_{n=0}^{\infty} (a)_n z^n, |z| < \rho$$

then,

$$D_z^\mu (z^{\lambda-1} f(z)) = \sum_{n=0}^{\infty} (a)_n D_z^{\mu} (z^{\lambda+n-1}) = \frac{\Gamma(\lambda)}{\Gamma(\lambda-\mu)} z^{\lambda-\mu-1} \sum_{n=0}^{\infty} (\lambda)_n (\mu)_n z^n$$

provided that $Re(\lambda) > 0, Re(\mu) < 0$, and $|z| < \rho$.

Some of the definition and notations used in the given manuscript are stated below:

Appell function of two variables defined by [1] are given as

$$F_1[a, b, b'; c, x, y] = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m (b')_n x^m y^n}{(c)_m (c')_n m! n!}$$

(1.4)

$$F_2[a, b, b'; c, c'; x, y] = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m (b')_n x^m y^n}{(c)_m (c')_n m! n!}$$

(1.5)

Generalization of Appell function of two variables by Khan M.A. and Abukhammash G.S. [2] are defined as

$$M_3(a, b, b', c, c'; d, d', e, e'; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m (b')_n (c)_m (c')_n x^m y^n}{(d)_m (d')_n (e)_m (e')_n m! n!}$$

(1.6)
\[ M_4(a, b', c', c'; d, e, e'; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_{m}(b')_{n}(c)_{m}(c')_{n}}{(d)_{m+n}(e)_{m}(e')_{n}} \frac{x^m y^n}{m! n!} \]  
(1.7)

\[ M_7(a, b, c, c'; d, e, e'; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}(c)_{m}(c')_{n}}{(d)_{m+n}(e)_{m}(e')_{n}} \frac{x^m y^n}{m! n!} \]  
(1.8)

\[ M_8(a, b, c, c'; d, e; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}(c)_{m}}{(d)_{m+n}(e)_{m+n}} \frac{x^m y^n}{m! n!} \]  
(1.9)

Lauricella [3] generalized the Appell double hypergeometric functions \( F_1, \ldots, F_4 \) to functions of \( n \) variables, but we use only two \( F_A^{(n)} \) and \( F_B^{(n)} \) are defined by

\[ F_A^{(n)}[a, b_1, \ldots, b_n; c_1, \ldots, c_n; x_1, \ldots, x_n] = \sum_{m_1, \ldots, m_n=0}^{\infty} \frac{(a)_{m_1+\ldots+m_n}(b_1)_{m_1} \ldots (b_n)_{m_n}}{(c_1)_{m_1} \ldots (c_n)_{m_n}} \frac{x_1^{m_1} \ldots x_n^{m_n}}{m_1! \ldots m_n!} \]  
(1.10)

\[ F_B^{(n)}[a, b_1, \ldots, b_n; c; x_1, \ldots, x_n] = \sum_{m_1, \ldots, m_n=0}^{\infty} \frac{(a)_{m_1+\ldots+m_n}(b_1)_{m_1} \ldots (b_n)_{m_n}}{m_1! \ldots m_n!} \frac{x_1^{m_1} \ldots x_n^{m_n}}{(c)_{m_1+\ldots+m_n}} \]  
(1.11)

In 1963, Pandey [6] established two interesting Horn’s type hypergeometric functions of three variables, while transforming Pochhammer’s double-loop contour integrals associated with the Lauricella’s functions \( F_A \) and \( F_B \) are given below:

\[ G_A[a, \beta, \beta'; \gamma; x, y, z] = \sum_{m,n,p=0}^{\infty} \frac{(a)_{n+p+m}(b)_{m+p}(\beta')_{n}}{(\gamma)_{n+p+m} m! n! p!} x^m y^n z^p \]  
(1.12)

\[ G_B[a, \beta, \beta_1, \beta_2; \gamma; x, y, z] = \sum_{m,n,p=0}^{\infty} \frac{(a)_{n+p+m}(b)_{m}(\beta_1)_{n}(\beta_2)_{p}}{(\gamma)_{n+p+m} m! n! p!} x^m y^n z^p \]  
(1.13)

In this manuscript we use the following fractional derivative formulas [9] to obtain the several class of generating functions.

\[ D_{x_1}^{a-\alpha} D_{x_2}^{a-\alpha} \left\{ x_1^{-\alpha} x_2^{-\alpha} \left[ 1 - \frac{\omega_1}{x_1} - \frac{\omega_2}{x_2} \right]^{-\alpha} \right\} = \frac{\Gamma(1-a) \Gamma(1-a')}{\Gamma(1-\alpha) \Gamma(1-\alpha')} x_1^{-\alpha} x_2^{-\alpha} F_2 \left[ \beta, \mu, \mu'; \alpha, \alpha'; x_1, x_2 \right] \]  
(1.14)

where, \( \left| \frac{\omega_1}{x_1} + \frac{\omega_2}{x_2} \right| < 1 \).

\[ D_{x_1}^{a-\alpha} D_{x_2}^{a-\alpha} D_{x_3}^{a-\alpha} \left\{ x_1^{-\alpha} x_2^{-\alpha} x_3^{-\alpha} \left[ 1 - \frac{\omega_1}{x_1} - \frac{\omega_2}{x_2} - \frac{\omega_3}{x_3} \right]^{-\alpha} \right\} = \frac{\Gamma(1-a) \Gamma(1-a) \Gamma(1-a)}{\Gamma(1-\alpha) \Gamma(1-\alpha) \Gamma(1-\alpha)} x_1^{-\alpha} x_2^{-\alpha} x_3^{-\alpha} \]  
(1.15)

\[ \times F_3 \left[ \beta, \mu_1, \mu_2, \mu_3; \alpha_1, \alpha_2, \alpha_3; x_1 x_2 x_3 \right] \]  

where, \( \left| \frac{\omega_1}{x_1 x_2} + \frac{\omega_2}{x_1 x_3} \right| + \left| \frac{\omega_3}{x_2 x_3} \right| < 1 \).

\[ D_{x_1}^{a-\alpha} D_{x_2}^{a-\alpha} D_{x_3}^{a-\alpha} \left\{ x_1^{-\alpha} x_2^{-\alpha} x_3^{-\alpha} \left[ 1 - \frac{\omega_1}{x_1 x_2 x_3} \right]^{-\alpha} \right\} = \frac{\Gamma(1-a) \Gamma(1-a) \Gamma(1-a)}{\Gamma(1-\alpha) \Gamma(1-\alpha) \Gamma(1-\alpha)} x_1^{-\alpha} x_2^{-\alpha} x_3^{-\alpha} M_7 \left[ \beta, \mu_1, \mu_2, \mu_3; \alpha_1, \alpha_2, \alpha_3; x_1 x_2 x_3 \right] \]  
(1.16)

where, \( \left| \frac{\omega_1}{x_1 x_2 x_3} + \frac{\omega_2}{x_1 x_3} \right| + \left| \frac{\omega_3}{x_2 x_3} \right| < 1 \).

\[ D_{x}^{a-\alpha} \left\{ x^{a} (1-x)^{-\alpha} \right\} \left( 1 - \frac{\omega_1}{1-x} \right)^{-\alpha} = \frac{\Gamma(1-a) \Gamma(1+a)}{\Gamma(1+\mu) \Gamma(1+\mu-\alpha)} x^{a} \]  
(1.17)

where, \( \text{Re}(\alpha) \geq 0, |x| < 1, |\omega_1 x + \frac{\omega_2}{1-x}| < 1 \).

\[ D_{x}^{a-\alpha} \left\{ x^{a} (1-x)^{-\alpha} \right\} \left( 1 - \frac{\omega_1}{1-x} \right)^{-\alpha} = \frac{\Gamma(1-a) \Gamma(1+a)}{\Gamma(1+\mu) \Gamma(1+\mu-\alpha)} x^{a} \]  
(1.18)
where, \( \text{Re}(\alpha) \geq 0, |x| < 1, \frac{\omega x}{1-x} < 1 \).

\[
D_x^{\alpha-\mu} \left\{ x^\alpha (1-x)^{-\beta} \left( 1 - \frac{\omega_1 x}{1-x} \right)^{\gamma} \left( 1 - \frac{\omega_2 x}{1-x} \right)^{\delta} \right\} = \frac{\Gamma(1+\alpha)}{\Gamma(1+\mu)} \Gamma^{(3)} \left[ 1 + \alpha; -\delta; \beta; \gamma \right] \Gamma(1-\beta) \Gamma(1-\gamma) \left[ 1 - \frac{\omega_1}{x} \right]_{\beta} \cdot \left[ 1 - \frac{\omega_2}{x} \right]_{\gamma} \left[ 1 - \frac{\omega_2}{x} \right]_{\delta}
\]

\( (1.19) \)

where, \( \text{Re}(\alpha) \geq 0, |x| < 1, \frac{\omega x}{1-x} < 1, \frac{\omega_2 x}{1-x} < 1 \).

\[
D_x^{\alpha-\mu} \left\{ x^\alpha (1-x)^{-\beta} \left( 1 - \omega_1 x \right)^{\gamma} \right\} = \frac{\Gamma(1+\alpha)}{\Gamma(1+\mu)} \Gamma^{(3)} \left[ 1 + \alpha; -\gamma; \beta; \gamma \right] \Gamma(1-\gamma) \left[ 1 - \frac{\omega_1}{x} \right]_{\gamma} \left[ 1 - \frac{\omega_2}{x} \right]_{\gamma} \left[ 1 - \frac{\omega_2}{x} \right]_{\gamma}
\]

\( (1.20) \)

where, \( \text{Re}(\alpha) \geq 0, |x| < 1, |\omega x| < 1, \frac{\omega_2 x}{1-x} < 1 \).

\[
D_x^{\alpha-\mu} \left\{ x^\alpha (1-x)^{-\beta} \left( 1 - \omega_1 x_1 x_2 \right)^{-\gamma} \right\} = \frac{\Gamma(1+\alpha)}{\Gamma(1+\mu)} \Gamma^{(3)} \left[ 1 + \alpha; -\gamma; \beta; \gamma \right] \Gamma(1-\gamma) \left[ 1 - \frac{\omega_1}{x_1 x_2} \right]_{\gamma} \left[ 1 - \frac{\omega_2}{x_1 x_2} \right]_{\gamma} \left[ 1 - \frac{\omega_2}{x_1 x_2} \right]_{\gamma}
\]

\( (1.21) \)

where, \( \frac{\omega_1}{x_1 x_2} < 1, \frac{\omega_2}{x_1 x_2} < 1 \).

\[
D_x^{\alpha-\mu} \left\{ x^\alpha \left( 1 - \omega_1 x \right)^{-\beta} \right\} = \frac{\Gamma(1+\alpha)}{\Gamma(1+\mu)} \Gamma^{(2)} \left[ 1 + \alpha; -\beta; \gamma \right] \Gamma(1-\beta) \left[ 1 - \frac{\omega_1}{x} \right]_{\beta} \left[ 1 - \frac{\omega_2}{x} \right]_{\beta} \left[ 1 - \frac{\omega_2}{x} \right]_{\beta}
\]

\( (1.22) \)

where, \( \text{Re}(\alpha) \geq 0, |\omega x| < 1, \frac{\omega_2 x}{1-x} < 1 \).

**II. LINEAR GENERATING FUNCTIONS**

Consider the elementary identity (see, [12], Sec. 5.2, Eq. 1),

\[
(1-x)^{-\alpha} = \Gamma(1-\alpha) \left[ 1 - \frac{x}{1-t} \right]^{-\alpha} \Gamma(1-t)^{-\alpha} (1-t)^{-\alpha} \left[ 1 - \frac{x}{1-t} \right]^{-\alpha}
\]

\( (2.1) \)

can be written as

\[
\sum_{n=0}^{\infty} \frac{(\lambda)^n}{n!} (1-x)^{-\alpha} t^n = (1-t)^{-\alpha} \left[ 1 - \frac{x}{1-t} \right]^{-\alpha} (1-t)^{\alpha} t^n, |t| < |1-x|.
\]

\( (2.2) \)
Generating functions of certain hypergeometric functions by means of fractional calculus

Replace \( x \) by \( \frac{\omega_1}{x_1} + \frac{\omega_2}{x_2} \) multiply both side of (2.2) by \( x_1^{-\alpha}x_2^{-\alpha'} \) and then apply the fractional derivative operator \( D^{\alpha}_{x_1}D^{\alpha'}_{x_2} \), to obtain

\[
\sum_{n=0}^{\infty} \frac{(\lambda)^n}{n!} D^{\alpha}_{x_1}D^{\alpha'}_{x_2} \left\{ x_1^{-\alpha}x_2^{-\alpha'} \left( 1 - \frac{\omega_1}{x_1} - \frac{\omega_2}{x_2} \right)^{-(\lambda+n)} \right\} t^n
\]

\[
= (1 - t)^{-\lambda}D^{\alpha}_{x_1}D^{\alpha'}_{x_2} \left\{ x_1^{-\alpha}x_2^{-\alpha'} \left( 1 - \frac{\omega_1}{x_1(1-t)} - \frac{\omega_2}{x_2(1-t)} \right)^{-\lambda} \right\}
\]

(2.3)

with the aid of the relation (1.14), equation (2.3) yield's the generating relation,

\[
\sum_{n=0}^{\infty} \frac{(\lambda)^n}{n!} F \left\{ \alpha_1, \alpha'_1, \alpha_2, \alpha'_2; \frac{\omega_1}{x_1}, \frac{\omega_2}{x_2} \right\} t^n
\]

\[
= (1 - t)^{-\lambda} F \left\{ \lambda, \mu_1, \mu'_1, \alpha'_1, \alpha'_2; \frac{\omega_1}{x_1(1-t)}, \frac{\omega_2}{x_2(1-t)} \right\}
\]

(2.4)

Similarly, the generalization of the generating function (2.4) can be obtained in the following form:

\[
\sum_{n=0}^{\infty} \frac{(\lambda)^n}{n!} P^{(n)} \left\{ \lambda, n, \mu_1, \mu_2, \mu_3; \alpha_1, \alpha_2, \alpha_3, \alpha_4; \frac{\omega_1}{x_1}, \frac{\omega_2}{x_2} \right\} t^n
\]

\[
= (1 - t)^{-\lambda} P^{(n)} \left\{ \lambda, \mu_1, \mu_2, \mu_3; \alpha_1, \alpha_2, \alpha_3, \alpha_4; \frac{\omega_1}{x_1(1-t)}, \frac{\omega_2}{x_2(1-t)} \right\}
\]

(2.5)

Replace \( x \) by \( \frac{\omega_1}{x_1} + \frac{\omega_2}{x_2} \), multiply both side of (2.2) by \( x_1^{-\alpha}x_2^{-\alpha'}x_3^{-\alpha''}x_4^{-\alpha'''} \) and then apply the fractional derivative operator \( D^{\alpha}_{x_1}D^{\alpha'}_{x_2}D^{\alpha''}_{x_3}D^{\alpha'''}_{x_4} \), to obtain

\[
\sum_{n=0}^{\infty} \frac{(\lambda)^n}{n!} D^{\alpha}_{x_1}D^{\alpha'}_{x_2}D^{\alpha''}_{x_3}D^{\alpha'''}_{x_4} \left\{ x_1^{-\alpha}x_2^{-\alpha'}x_3^{-\alpha''}x_4^{-\alpha'''} \left( 1 - \frac{\omega_1}{x_1x_2} - \frac{\omega_2}{x_3x_4} \right)^{-(\lambda+n)} \right\} t^n
\]

\[
= (1 - t)^{-\lambda}D^{\alpha}_{x_1}D^{\alpha'}_{x_2}D^{\alpha''}_{x_3}D^{\alpha'''}_{x_4} \left\{ x_1^{-\alpha}x_2^{-\alpha'}x_3^{-\alpha''}x_4^{-\alpha'''} \left( 1 - \frac{\omega_1}{x_1x_2(1-t)} - \frac{\omega_2}{x_3x_4(1-t)} \right)^{-\lambda} \right\}
\]

(2.6)

with the aid of the relation (1.15), equation (2.6) yield's the generating relation,

\[
\sum_{n=0}^{\infty} \frac{(\lambda)^n}{n!} M_3 \left\{ \lambda, n, \mu_1, \mu_2, \mu_3, \mu_4; \alpha_1, \alpha_2, \alpha_3, \alpha_4; \frac{\omega_1}{x_1x_2}, \frac{\omega_2}{x_3x_4} \right\} t^n
\]

\[
= (1 - t)^{-\lambda} M_3 \left\{ \lambda, \mu_1, \mu_2, \mu_3, \mu_4; \alpha_1, \alpha_2, \alpha_3, \alpha_4; \frac{\omega_1}{x_1x_2(1-t)}, \frac{\omega_2}{x_3x_4(1-t)} \right\}
\]

(2.7)

Replace \( x \) by \( \frac{\omega_1}{x_1} + \frac{\omega_2}{x_2} \), multiply both side of (2.2) by \( x_1^{-\alpha}x_2^{-\alpha'}x_3^{-\alpha''}x_4^{-\alpha'''} \) and then apply the fractional derivative operator \( D^{\alpha}_{x_1}D^{\alpha'}_{x_2}D^{\alpha''}_{x_3}D^{\alpha'''}_{x_4} \), to obtain

\[
\sum_{n=0}^{\infty} \frac{(\lambda)^n}{n!} D^{\alpha}_{x_1}D^{\alpha'}_{x_2}D^{\alpha''}_{x_3}D^{\alpha'''}_{x_4} \left\{ x_1^{-\alpha}x_2^{-\alpha'}x_3^{-\alpha''}x_4^{-\alpha'''} \left( 1 - \frac{\omega_1}{x_1x_2} - \frac{\omega_2}{x_3x_4} \right)^{-(\lambda+n)} \right\} t^n
\]

\[
= (1 - t)^{-\lambda}D^{\alpha}_{x_1}D^{\alpha'}_{x_2}D^{\alpha''}_{x_3}D^{\alpha'''}_{x_4} \left\{ x_1^{-\alpha}x_2^{-\alpha'}x_3^{-\alpha''}x_4^{-\alpha'''} \left( 1 - \frac{\omega_1}{x_1x_2(1-t)} - \frac{\omega_2}{x_3x_4(1-t)} \right)^{-\lambda} \right\}
\]

(2.8)

with the aid of the relation (1.16), equation (2.8) yield's the generating relation,

\[
\sum_{n=0}^{\infty} \frac{(\lambda)^n}{n!} M_7 \left\{ \lambda, n, \mu_1, \mu_2, \mu_3, \mu_4; \alpha_1, \alpha_2, \alpha_3; \frac{\omega_1}{x_1x_2}, \frac{\omega_2}{x_1x_3} \right\} t^n
\]

\[
= (1 - t)^{-\lambda} M_7 \left\{ \lambda, \mu_1, \mu_2, \mu_3, \mu_4; \alpha_1, \alpha_2, \alpha_3; \frac{\omega_1}{x_1x_2(1-t)}, \frac{\omega_2}{x_1x_3(1-t)} \right\}
\]

(2.9)

Replace \( x \) by \( \omega_1x + \omega_2 \), multiply both side of (2.2) by \( x^n(1-x)^{-\beta} \) and then on applying the fractional derivative operator \( D^{\alpha}_{x} \), to obtain

\[
\sum_{n=0}^{\infty} \frac{(\lambda)^n}{n!} D^{\alpha}_{x} \left\{ x^n(1-x)^{-\beta} \left( 1 - \omega_1x - \frac{\omega_2}{1-x} \right)^{-(\lambda+n)} \right\} t^n
\]

\[
= (1 - t)^{-\lambda} D^{\alpha}_{x} \left\{ x^n(1-x)^{-\beta} \left( 1 - \frac{\omega_1x}{1-t} - \frac{\omega_2}{1-x} \right)^{-\lambda} \right\}
\]

(2.10)

with the aid of the relation (1.17), equation (2.10) yield's the generating relation,

\[
\sum_{n=0}^{\infty} \frac{(\lambda)^n}{n!} H_\beta \left\{ \lambda, \omega, 1 + \alpha; \beta, 1 + \mu; \omega_1, \omega_2, 1 \right\} t^n
\]
Generating functions of certain hypergeometric functions by means of fractional calculus

\[ (1 - t)^{-\frac{\omega_1}{\omega_2}} H_A[\beta, \lambda, 1 + \alpha; \beta, 1 + \mu; \frac{\omega_2}{\omega_1}, \frac{\omega_1 x}{(1 - t)}] \]

(2.11)

where hypergeometric function \( H_A \) is defined by Srivastava [10] as

\[
H_A[\alpha, \beta; \gamma; \gamma'; x, y, z] = \sum_{m,n,p=0}^{\infty} \left( \frac{\alpha}{m+n+p} \frac{\beta}{m+n} \frac{\beta'}{n+p} \right) \frac{x^m y^n z^p}{(\gamma) (\gamma')_{n+p} m! n! p!}
\]

(2.12)

The analysis used to obtain the relation (2.11) is further employed to obtain linear generating function by using the relation (1.18), is given below:

\[
\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} F_1 \left[ \frac{1 + \alpha, \lambda + n, 1 + \mu - \beta; 1 + \mu; \omega x}{1 - x; x, x - 1} \right] t^n
\]

(2.13)

Further, in (2.2), replace \( x \) by \( \frac{\omega_1 x}{1 - x'} \) \( t \) by \( t_1, t_2 \) and \( \lambda \) by \( \lambda_1, \lambda_2 \) respectively. Then multiply the two equation, to obtain

\[
\sum_{m,n=0}^{\infty} \left( \frac{(\lambda_1)_m (\lambda_2)_n}{m! n!} \right) \left( 1 - \frac{\omega_1 x}{1 - x'} \right)^{(1 + m)} \left( 1 - \frac{\omega_2 x}{1 - x'} \right)^{(1 + n)} (t_1)^m (t_2)^n
\]

(2.14)

Now multiply both sides of (2.14) by \( x^\beta (1 - x)^{-\beta} \) and then by appealing the fractional derivative operator \( D_x^\alpha \) on both sides and using (1.19), one obtains the double sum generating relation as

\[
\sum_{m,n=0}^{\infty} \left( \frac{(\lambda_1)_m (\lambda_2)_n}{m! n!} \right) (t_1)^m (t_2)^n F_0[\beta](1 + \alpha, \lambda_1 + m, \lambda_2 + n, 1 + \mu - \beta; 1 + \mu; \omega_1 x, \omega_2 x, x)
\]

(2.15)

where \( F_0[\beta] \) is defined by (1.11) at \( n = 3 \).

Further, we adopt the analysis similar to (2.15) and use the relations (1.20), (1.21), respectively, yield's the following double sum generating functions as follows:

\[
\sum_{m,n=0}^{\infty} \left( \frac{(\lambda_1)_m (\lambda_2)_n}{m! n!} \right) (t_1)^m (t_2)^n F_M[\lambda_2 + n, 1 + \alpha, 1 + \alpha, \beta, \lambda_1 + m, \beta; 1 + \mu, 1 + \mu; \omega_2, \omega_1 x, x]
\]

(2.16)

where \( F_M \) is defined by (8) as

\[
F_M[\alpha_1, \alpha_2, \alpha_3; \beta_1, \beta_2, \beta_3; \gamma_1, \gamma_2, \gamma_3; x, y, z] = \sum_{m,n,p=0}^{\infty} \left( \frac{(\alpha_1)_m (\alpha_2)_n (\beta_1)_m (\beta_2)_n (\beta_3)_p}{(\gamma_1)_m (\gamma_2)_n (\gamma_3)_p} \right) \frac{x^m y^n z^p}{m! n! p!}
\]

(2.17)

where the general triple hypergeometric series \( F_M[\alpha, \beta, \gamma; x, y, z] \) is defined by [11] as

\[
F^3[\alpha; \beta; \gamma; x, y, z] = F^3 \left[ \frac{\left( \begin{array}{c} 1 + \alpha \vdots - \beta; - \lambda_1; m; \lambda_2; \lambda_3; - \mu_1; \vdots - \mu_2; \vdots \omega_1 x, \omega_2 x, x \end{array} \right)}{1 + \mu \vdots - \beta; - \lambda_1; \lambda_2; \lambda_3; - \mu_1; \vdots - \mu_2; \vdots \omega_1 x, \omega_2 x, x} \right]
\]

(2.18)

where the general triple hypergeometric series \( F_M[\alpha, \beta, \gamma; x, y, z] \) is defined by [11] as

\[
F^3[\alpha; \beta; \gamma; x, y, z] = F^3 \left\{ \frac{(\alpha)_m (\beta)_n (\gamma)_p}{m! n! p!} \right\}
\]

(2.19)

where, for convenience,

\[
\Lambda(m, n, p) = \frac{\Pi_{j=1}^{A} (a_j)_m + n + p \Pi_{j=1}^{B} (b_j)_m + n \Pi_{j=1}^{C} (b'_j)_n + p \Pi_{j=1}^{n} (b''_j)_p + m}{\Pi_{j=1}^{A} (b_j)_m + n \Pi_{j=1}^{C} (a_j)_m + n \Pi_{j=1}^{n} (b'_j)_p + m} \times \frac{\Pi_{j=1}^{A} (a'_j)_m \Pi_{j=1}^{C} (a''_j)_n \Pi_{j=1}^{n} (a'''_j)_p}{\Pi_{j=1}^{A} (h_j)_m \Pi_{j=1}^{C} (h'_j)_n \Pi_{j=1}^{n} (h''_j)_p}
\]
Again, in (2.2), replace $x$ by $x^\frac{\alpha_1}{x_1} \frac{\alpha_2}{x_2} t$ by $t_1$, $t_2$ and $\lambda$ by $\lambda_1, \lambda_2$ respectively. Then multiply the two equation, to obtain

$$
\sum_{n=0}^{\infty} \frac{(\lambda_1)_m (\lambda_2)_n}{m! n!} \left(1 - \frac{\omega_1}{x_1 x_2}\right)^{-(\lambda_1+m)} \left(1 - \frac{\omega_2}{x_1 x_3}\right)^{-(\lambda_2+n)} (t_1)^m (t_2)^n
$$

$$
= (1 - t_1)^{-\lambda_1} (1 - t_2)^{-\lambda_2} \sum_{n=0}^{\infty} \frac{(\lambda_1)_m (\lambda_2)_n}{m! n!} \left(1 - \frac{\omega_1}{x_1 x_2}\right)^{-(\lambda_1+m)} \left(1 - \frac{\omega_2}{x_1 x_3}\right)^{-(\lambda_2+n)} (t_1)^m (t_2)^n
$$

(2.20)

Now multiply both sides of (2.20) by $x_1^{\omega_1} x_2^{\omega_2} x_3^{\omega_3}$ and then by appealing the fractional derivative operator $D_1^{\mu_1} D_2^{\mu_2} D_3^{\mu_3}$ on both sides and using (1.22), one obtains the double sum generating relation as

$$
\sum_{n=0}^{\infty} \frac{(\lambda_1)_m (\lambda_2)_n}{m! n!} M_4 \left[\mu_1, \lambda_1 + m, \lambda_2 + n, \mu_2, \lambda_3; \alpha_1, \alpha_2, \alpha_3; \frac{\omega_1}{x_1 x_2}, \frac{\omega_2}{x_1 x_3}, \frac{\omega_3}{x_1 x_3} \right] (t_1)^m (t_2)^n
$$

$$
= (1 - t_1)^{-\lambda_1} (1 - t_2)^{-\lambda_2} M_4 \left[\mu_1, \lambda_1, \lambda_2, \mu_2, \lambda_3; \alpha_1, \alpha_2, \alpha_3; \frac{\omega_1}{x_1 x_2}, \frac{\omega_2}{x_1 x_3}, \frac{\omega_3}{x_1 x_3} \right] (1 - t_1)^{\lambda_1} (1 - t_2)^{\lambda_2}
$$

(2.21)

Also, in (2.2), replace $x$ by $1 - \omega_1 x_1 x_2, 1 - \omega_1 x_1 x_2, t$ by $t_1, t_2$ and $\lambda$ by $\lambda_1, \lambda_2$ respectively. Then multiply the two equation, to obtain

$$
\sum_{n=0}^{\infty} \frac{(\lambda_1)_m (\lambda_2)_n}{m! n!} (1 - \omega_1 x_1 x_2)^{-(\lambda_1+m)} (1 - \omega_2 x_1 x_3)^{-(\lambda_2+n)} (t_1)^m (t_2)^n
$$

$$
= (1 - t_1)^{-\lambda_1} (1 - t_2)^{-\lambda_2} \sum_{n=0}^{\infty} \frac{(\lambda_1)_m (\lambda_2)_n}{m! n!} (1 - \omega_1 x_1 x_2)^{-(\lambda_1+m)} (1 - \omega_2 x_1 x_3)^{-(\lambda_2+n)} (t_1)^m (t_2)^n
$$

(2.22)

Now multiply both sides of (2.22) by $x_1^{\omega_1} x_2^{\omega_2}$ and then by appealing the fractional derivative operator $D_1^{\mu_1} D_2^{\mu_2}$ on both sides and using (1.23), one obtains the double sum generating relation as

$$
\sum_{n=0}^{\infty} \frac{(\lambda_1)_m (\lambda_2)_n}{m! n!} (t_1)^m (t_2)^n M_4 \left[1 + \alpha_1, 1 + \alpha_2, \lambda_1 + m, \lambda_2 + n; 1 + \mu_1, 1 + \mu_2; \omega_1 x_1 x_2, \omega_2 x_1 x_3 \right]
$$

$$
= (1 - t_1)^{-\lambda_1} (1 - t_2)^{-\lambda_2} M_4 \left[1 + \alpha_1, 1 + \alpha_2, \lambda_1, \lambda_2; 1 + \mu_1, 1 + \mu_2; \omega_1 x_1 x_2, \omega_2 x_1 x_3 \right]
$$

(2.23)

Also, in (2.2), replace $x$ by $x \omega_1, x \omega_2 + \frac{\omega_3}{x}, t$ by $t_1, t_2$ and $\lambda$ by $\lambda_1, \lambda_2$ respectively. Then multiply the two equation, to obtain

$$
\sum_{n=0}^{\infty} \frac{(\lambda_1)_m (\lambda_2)_n}{m! n!} (1 - \omega_1 x)^{-(\lambda_1+m)} (1 - \omega_2 x - \frac{\omega_3}{x})^{-(\lambda_2+n)} (t_1)^m (t_2)^n
$$

$$
= (1 - t_1)^{-\lambda_1} (1 - t_2)^{-\lambda_2} \sum_{n=0}^{\infty} \frac{(\lambda_1)_m (\lambda_2)_n}{m! n!} (1 - \omega_1 x)^{-(\lambda_1+m)} (1 - \omega_2 x - \frac{\omega_3}{x})^{-(\lambda_2+n)} (t_1)^m (t_2)^n
$$

(2.24)

Now multiply both sides of (2.24) by $x^\alpha$ and then by appealing the fractional derivative operator $D_x^{\alpha-\mu}$ on both sides and using (1.24), one obtains the double sum generating relation as

$$
\sum_{n=0}^{\infty} \frac{(\lambda_1)_m (\lambda_2)_n}{m! n!} G_d \left[1 + \alpha, \lambda_1, \lambda_2 + n, \lambda_1 + m; 1 + \mu; \omega_1 x, \omega_2 x, \omega_3 \right] (t_1)^m (t_2)^n
$$

$$
= (1 - t_1)^{-\lambda_1} (1 - t_2)^{-\lambda_2} G_d \left[1 + \alpha, \lambda_1, \lambda_2; 1 + \mu; \omega_1 x, \omega_2 x, \omega_3 \right]
$$

(2.25)

Also, in (2.2), replace $x$ by $\frac{\omega_1}{x}, \frac{\omega_2}{x}, t$ by $t_1, t_2$ and $\lambda$ by $\lambda_1, \lambda_2$ respectively. Then multiply the two equation, to obtain

$$
\sum_{n=0}^{\infty} \frac{(\lambda_1)_m (\lambda_2)_n}{m! n!} (1 - \omega_1 x)^{-(\lambda_1+m)} (1 - \omega_2 x)^{-(\lambda_2+n)} (t_1)^m (t_2)^n
$$

$$
= (1 - t_1)^{-\lambda_1} (1 - t_2)^{-\lambda_2} \sum_{n=0}^{\infty} \frac{(\lambda_1)_m (\lambda_2)_n}{m! n!} (1 - \omega_1 x)^{-(\lambda_1+m)} (1 - \omega_2 x)^{-(\lambda_2+n)} (t_1)^m (t_2)^n
$$

(2.26)

Now multiply both sides of (2.26) by $x^{-\alpha}$ and then by appealing the fractional derivative operator $D_x^\alpha-\mu$ on both sides and using (1.25), one obtains the double sum generating relation as

$$
\sum_{m,n=0}^{\infty} \frac{(\lambda_1)_m (\lambda_2)_n}{m! n!} P_1 \left[\mu, \lambda_1 + m, \lambda_2 + n; \omega_1, \omega_2 \right] (t_1)^m (t_2)^n
$$

www.ijceronline.com Open Access Journal Page 45
Generating functions of certain hypergeometric functions by means of fractional calculus

\[ (1 - t_1)^{-\lambda_1} (1 - t_2)^{-\lambda_2} F_1 \left[ \frac{\omega_1}{x(1 - t_1)}, \frac{\omega_2}{x(1 - t_2)} \right] \]

(2.27)

Similarly, the generalization of the generating function (2.27) can be obtained in the following form:

\[ \sum_{m_n,p=0}^{\infty} \frac{(-1)^m}{m! n! p!} (1 - \omega_1 x)^{-(\lambda_1 + m)} (1 - \omega_2 x)^{-(\lambda_2 + n)} \frac{\omega_3}{x(1 - t_3)} \] \[ x F_1 \left[ \frac{\omega_1}{(1 - t_1)}, \frac{\omega_3}{(1 - t_3)} \right] \frac{\omega_2}{x(1 - t_2)} \] \[ \frac{\omega_3}{x(1 - t_3)} \]

(2.28)

Replace \( x \) by \( \omega_1 x, \omega_2 x, \frac{\omega_3}{x} \) by \( t_1, t_2, t_3 \) and \( \lambda \) by \( \lambda_1, \lambda_2, \lambda_3 \) in (2.2) respectively. Then multiply the three equations, to obtain

\[ \sum_{m,n,p=0}^{\infty} \frac{(\lambda_1)_m (\lambda_2)_n (\lambda_3)_p}{m! n! p!} x F_1 \left[ \frac{\omega_1}{x}, \frac{\omega_3}{x}, \frac{\omega_2}{x} \right] \] \[ \frac{t_1^m (t_2)^n (t_3)^p}{(1 - t_1)(1 - t_2)(1 - t_3)} \] \[ \frac{\omega_3}{x(1 - t_3)} \]

(2.29)

Now multiply both sides of (2.29) by \( x^n \) and then by appealing the fractional derivative operator \( D_x^{m-n} \) on both sides and using (1.26), one obtains the triple sum generating relation as

\[ \sum_{m,n,p=0}^{\infty} \frac{(\lambda_1)_m (\lambda_2)_n (\lambda_3)_p}{n! m! p!} \frac{t_1^m (t_2)^n (t_3)^p}{(1 - t_1)(1 - t_2)(1 - t_3)} \] \[ x F_1 \left[ \frac{\omega_1}{x(1 - t_3)}, \frac{\omega_2}{x(1 - t_2)}, \frac{\omega_3}{x(1 - t_1)} \right] \] \[ \frac{\omega_3}{x(1 - t_3)} \]

(2.30)

Further, we adopt the analysis similar to (2.30) and use the relations (1.27), we obtain the following triple sum generating function as follows:

\[ \sum_{m,n,p=0}^{\infty} \frac{(\lambda_1)_m (\lambda_2)_n (\lambda_3)_p}{n! m! p!} \frac{t_1^m (t_2)^n (t_3)^p}{(1 - t_1)(1 - t_2)(1 - t_3)} \] \[ x F_1 \left[ \frac{\omega_1}{x(1 - t_3)}, \frac{\omega_2}{x(1 - t_2)}, \frac{\omega_3}{x(1 - t_1)} \right] \] \[ \frac{\omega_3}{x(1 - t_3)} \]

(2.31)

III. SPECIAL CASES

In this section, we mention some special cases of our previous results as given below:

On putting, \( \omega_2 = 0 \) and replacing \( \frac{\omega_1}{x} \) by \( z \) in (2.4), one obtains

\[ \sum_{n=0}^{\infty} \frac{(\lambda_n)_n}{n!} F_1 \left[ \frac{\lambda + n, \mu_1}{\alpha}; z \right] t^n = (1 - t)^{-\frac{1}{\alpha}} F_1 \left[ \frac{\lambda}{\alpha}; \frac{z}{1 - t} \right] \]

(3.1)

is the known result (see, [12], p.292, Eq.6).

Now, if on putting \( \alpha_2 = \mu_2, \alpha_4 = \mu_4, \omega_1 = x_2 \omega_1, \omega_2 = \frac{x_3 x_4 \omega_2}{x_2} \), in (2.7), one obtains the relation (2.4), which on further with usual replacement yield (3.1).

For \( \omega_2 = 0 \), in (2.11) and on replacing \( \omega_1 \) by \( \frac{x}{z} \), one obtains

\[ \sum_{n=0}^{\infty} \frac{(\lambda_n)_n}{n!} F_1 \left[ \frac{1 + \alpha, \lambda + n, \beta}{\alpha}; \frac{1 + \mu; x}{z} \right] t^n = (1 - t)^{-\frac{1}{\alpha}} F_1 \left[ \frac{1 + \alpha, \lambda, \beta}{\alpha}; \frac{1 + \mu; \frac{z}{1 - t} \alpha; \frac{x}{z}} \right] \]

(3.2)

which for \( x \to 0 \) reduces to known result ([12], p.292, Eq.6).

REFERENCES

Generating functions of certain hypergeometric functions by means of fractional calculus


