Approximation To $L^p$ Integrable Functions By Gamma Type Operators

V. K. Gupta, S. K. Tiwari, Yogita Parmar

1V. K. Gupta, Department of Mathematics, Govt. Madhav Science College Ujjain (M.P.)
2S. K. Tiwari Department of Mathematics, School of Studies in mathematics, Vikram University, Ujjain (M.P.)
3Yogita Parmar Research Scholar, School of Studies in mathematics, Vikram University, Ujjain (M.P.)

Corresponding Author: V. K. Gupta,

INTRODUCTION

A Korovkin type theorem for linear positive operators acting from $L^p(a,b)$ into $L^p(a,b)$ was studied in [7] and then some new result in this direction were established. Ditzian and Ivanov [5] studied Bernstein type operators and their derivatives in $L^p(0,1)$ spaces and order approximation of these operators by using the K-functional of Peetre. Direct theorems for linear combination of Szasz-Beta type operators which defined by Gupta et all [8] in $L^p$-approximation on positive semi axis obtained by Mahewshwari [20].

Our aim is to study approximation properties of $B_m(f;x)$ operators by means of Korovkin’s theorem in $L^p$-spaces and we get upper bound for that by using the K-functional of Peetre. In this paper we use the result of A. Izgi [12].

Keywords: $L^p$- approximation, Gamma type operators, K-functional of Peetre, Order of approximation in $L^p$-spaces.

ABSTRACT

In this paper we studied the following modification of Gamma operators which are first introduced in [9] (see [19], [21] and [9] respectively)

$$B_m(f;x) = \int_0^\infty K_m(x,t)f(t)dt$$

where

$$K_m(x,t) = \frac{(2m+3)!}{m!(m+2)!} \frac{t^m x^{m+3}}{(x+t)^{2m+4}} f(t), t \in (0,\infty),$$

and the approximation properties of these operators. We give approximation of $B_m(f;x)$ in $L^p$-spaces and we get upper bound for that by using the K-functional of Peetre. In this paper we use the result of A. Izgi [12].

Keywords: $L^p$- approximation, Gamma type operators, K-functional of Peetre, Order of approximation in $L^p$-spaces.

INTRODUCTION

A Korovkin type theorem for linear positive operators acting from $L^p(a,b)$ into $L^p(a,b)$ was studied in [7] and then some new result in this direction were established. Ditzian and Ivanov [5] studied Bernstein type operators and their derivatives in $L^p(0,1)$ spaces and order approximation of these operator by using the K-functional of Peetre. Direct theorems for linear combination of Szasz-Beta type operators which defined by Gupta et all [8] in $L^p$-approximation on positive semi axis obtained by Mahewshwari [20].

Our aim is to study approximation properties of $B_m(f;x)$ operators by means of Korovkin’s theorem in $L^p$-spaces and we get upper bound for that by using the K-functional of Peetre. In this paper we use the result of A. Izgi [12].

Keywords: $L^p$- approximation, Gamma type operators, K-functional of Peetre, Order of approximation in $L^p$-spaces.

II. PRELEMINARIES

The following operators given by Izgi and Buyukyazicy [9].

$$B_m(f;x) = \frac{(2m+3)!}{m!(m+2)!} \frac{t^m x^{m+3}}{(x+t)^{2m+4}} f(t), x > 0$$

If we choose

$$K_m(x,t) = \frac{(2m+3)!}{m!(m+2)!} \frac{x^{m+3} t^m}{(x+t)^{2m+4}} f(t), t \in (0,\infty),$$

We can write $B_m(f;x)$ as the following form:

$$B_m(x,t) = \int_0^\infty K_m(x,t)f(t)dt.$$
In [14] it was studied the rate of pointwise convergence of the operators $B_m(x, t)$ on the set of functions with bounded variation. These operators for bivariate functions in the weighted spaces with the following operators studied by Izgi. A.[10].

\begin{equation}
B_{m,n}(f(r,s)x,y) = \int_{0}^{\infty} \int_{0}^{\infty} K_m(x,r)K_m(y,r)f(r,s)drds
\end{equation}

and also studied $B_m(f; x)$ for Voronoskaya type asymptotic approximation by Izgi. A in [11].

Now we introduce some notations which will be used in main result.

We denote by $C_b(0, \infty)$ the class of continuous and bounded functions on $(0, \infty)$by $BC(0, \infty)$the spaces of all absolutely continuous functions on $(0, \infty)$ and by $L^p_{p}(0, \infty)$, a subspaces of $L^p(0, \infty)$ such that

$$L^p_{p}(0, \infty) = \{ f \in L^p(0, \infty); f' \in BC(0, \infty), f'' \in L^p(0, \infty) \ for \ 1 \leq p \leq \infty \}.$$ 

The norm on the spaces$L^p_{p}(0, \infty)$ can be defined as

$$\| g \|_{L^p_{p}} = \| g \|_{L^p} + \| g^{00} \|_{L^p}$$

Or equivalently

$$\| g \|_{L^p_{p}} = \sum_{k=0}^{2} \| g^{(k)} \|_{L^p}$$

$$= \sum_{k=0}^{2} (\int |g^{(k)}(t)|^p dt)^{1/p}$$

$$= \| g \|_{L^p} + \| g^0 \|_{L^p} + \| g^{00} \|_{L^p}$$

We consider also following K-functional of Peetre [24].

$$K_p(f; \delta) = \inf_{g \in L^p_{p}(0, \infty)} \{ \| f - g \|_{L^p(0, \infty)} + \delta \| g \|_{L^p_{p}(0, \infty)} \}, \delta \geq 0$$

For $f \in L^p((0, \infty))$, using Theorem 2, we have $\lim_{\delta \to 0} K_p(f; \delta) = 0$. Therefore the K-functional gives the degree of approximation of a function $f \in L^p(0, B)$ by smoother functions $g \in L^p_{p}((0, B))$.

Remember that the second order integral modulus of smoothness is given by

$$\vartheta_{2,p}(f; \delta) = \sup_{0 \leq \lambda \leq \delta} \| f(x + h) - 2f(x) + f(x - h) \|_{L^p(0, \infty)} (h)$$

For an $f \in L^p(0, B)$, where $L_b$ indicates that the $L^p$-norm is taken over the interval $[h, B - h]$.

It is also known that there are constants $c_1 > 0, c_2 > 0$, independent of $f$ and $p$ such that

$$c_1 \vartheta_{2,p} \left( f; \delta^{1/2} \right) \leq K_p(f; \delta) \leq \min(1, \delta) \| f \|_{L^p(0, \infty)} + 2c_2 \vartheta_{2,p} \left( f; \delta^{1/2} \right)$$

We need the following properties of $B_m(f; x)$ which where shown in [9]:

For any $p \in \mathbb{N}$, $p \leq m + 2$

$$B_m(t^p; x) = \frac{(m + p)! (m + 2 - p)!}{m! (m + 2)!} \frac{1}{x^p}$$
Approximation To $L^p$ Interable Functions By Gamma Type Operators

It follows from (4) that
\[ B_m(1; x) = 1 \quad (5) \]
\[ B_m(t; x) = x - \frac{x}{m + 2} \quad (6) \]
\[ B_m(t^2; x) = x^2 \quad (7) \]

The following equalities are hold by (5), (6) and (7):
\[ B_m((t - x)^2; x) = \frac{2}{m + 2} x^2 \quad (8) \]
\[ \sup_{x \in (0, B]} B_m((t - x)^2; x) = \frac{2}{m + 2} B^2 \quad (9) \]

**Theorem 1**: Let \( f \in C_{b}(0, \infty) \). Then for a real number \( B > 0 \), the limit relation
\[ \lim_{m \to \infty} B_m(f; x) = f(x) \]
holds uniformly on \((0, B]\).

**Proof**: Using (6), (7) and (8) we see that:
\[ \|B_m(1; x) - 1\|_{C(0,B]} = 0 \]
\[ \|B_m(t; x) - x\|_{C(0,B]} = \max_{x \in (0,B]} \frac{x}{m + 2} \leq \frac{B}{m + 2} \to 0, \quad (m \to \infty) \]

by P.P. Korovkin theorem [17], the proof of Theorem 1 is completed.

**III. MAIN RESULTS FOR THE APPROXIMATION IN LP-SPACES**

In this section, we prove theorems of Korovkin type for approximation in the norm of the space \( L^p(0, B] \), \( 1 \leq p \leq \infty \), of integrable functions whose first derivatives belong to the class absolutely continuous functions \((0, \infty)\) and second derivatives belong to the class \( L^p(0, \infty) \) and we will give a rate of convergence using the K-functional of Peetre [24];

It is easily to see that,
\[ \int_0^\infty K_m(x,t)dt = 1, \quad \text{and} \quad \int_0^\infty K_m(x,t)dx = \frac{m + 3}{m} \leq 4 \quad (10) \]

Thus \( B_m(f;x) \) exists for all \( f \in L^p(0, \infty) \) and for every fixed \( m \). (see [18] cf. 31 Theorem of Orlicz). According to Lusinos theorem, if \( f \in L^p(0, B] \) then there exists a function \( g \in C(0,B] \) such that for any \( \varepsilon > 0 \)
\[ \varphi((|x|f(x) \neq g(x))) = \varepsilon, \quad (11) \]

Now we give the following theorem for the approximation in the \( L^p \) spaces, \( p \geq 1 \).

**Theorem 2**: Let \( f \in L^p(0, \infty) \) and \( B \) be a fixed derivative point in \((0, \infty)\) such that the condition,
\[ \frac{|f(t) - f(x)|}{|t - x|} \leq M, \quad x \in (0, B], t \in (B, \infty) \quad (12) \]
holds with the constant M. Then
\[ \|B_m f - f\|_{L^p(0,B]} \to 0, \quad (m \to \infty). \]

**Proof:** By (11)
\[ \|g - f\|_{L^p(0,B]} < \varepsilon \] holds.

From Theorem 1, \( \|B_m g - g\|_{C[0,B]} \to 0 \) \((m \to \infty)\). Thus for \( \varepsilon > 0 \) there exists an \( m_0 \in N \) such that for all \( m > m_0 \),
\[ \|B_m g - g\|_{C[0,B]} < \varepsilon. \]

Now we can write that
\[
B_m (f:x) - f(x) = \int_0^B K_m(x,t) (f(t) - f(x)) dt
\]
\[
= \int_0^B K_m(x,t) (f(t) - f(x)) dt + \int_B^\infty K_m(x,t) (f(t) - f(x)) dt
\]
\[
= E_1(x) + E_2(x)
\] (14)
\[
|E_1(x)| \leq \int_0^B K_m(x,t) (f(t) - f(x)) dt
\]
\[
\leq \int_0^B K_m(x,t) |f(t) - g(t)| dt + \int_0^B K_m(x,t) |g(t) - g(x)| dt + \int_0^B K_m(x,t) |g(x) - f(x)| dt
\]
\[
\leq E_{11}(x) + E_{12}(x) + E_{13}(x)
\] (15)

For sufficiently large \( m \) by (13)
\[ \|E_{11}(x)\|_{L^p(0,B]} \leq \left( \int_0^B |f(t) - g(t)|^p dt \right)^{\frac{1}{p}} < \varepsilon. \] (16)

Now, we evaluate \( \|E_{12}(x)\|_{L^p(0,B]} \). Since \( g \) is a continuous function in \((0, B]\) we can write well known inequality
\[ |g(t) - g(x)| < \varepsilon + \frac{2M_1(t-x)^2}{\delta^2}, \]
where \( \delta > 0 \) and \( M_1 \) constant such that \( |g(x)| < M_1 \). Then
\[ E_{12}(x) = \int_0^B K_m(x,t) |f(t) - g(t)| dt \leq \varepsilon \int_0^B K(x,t) dt + \frac{2M_1}{\delta^2} \int_0^\infty (t-x)^2 K(x,t) dt \]
and by (9)
\[ E_{12}(x) < \varepsilon + \frac{2M_1}{\delta^2} \frac{2}{2+m} B^2 \]
Since $\frac{2B^2}{2+m} \to 0$ as $m \to \infty$, for a large $m$

$$\|E_{12}(x)\|_{L^p(0,B)} < C\varepsilon, \quad (17)$$

where $C$ is a positive constant. If $|t - x| < \varphi$ then $|g(t) - g(x)| < \varepsilon$, hence

$$\int_0^\infty K_m(x,t)|g(t) - g(x)| dt < \varepsilon.$$ 

By (14) we have

$$\|E_{13}(x)\|_{L^p(0,B)} < \varepsilon. \quad (18)$$

Thus for large $m$,

$$\|E_1(x)\|_{L^p(0,B)} < \varepsilon. \quad (19)$$

Consider $E_2(x)$ using the condition (12) and Holder’s inequality, we get

$$|E_2(x)| \leq \int_B^\infty |K_m(x,t) (f(t) - f(x))| dt \leq M \int_B^\infty K_m(x,t)|t - x| dt \leq M \sqrt{\int_B^\infty K_m(x,t) dt} \sqrt{\int_B^\infty K_m(x,t)|t - x|^2 dt} \leq M \sqrt{\delta_m},$$

where $\delta_m = \frac{2B^2}{m+2}$ (see (9)).

Thus,

$$\|E_2(x)\|_{L^p(0,B)} \leq M \sqrt{\delta_m B^\frac{1}{p}} \quad (20)$$

and therefore (13), (19) and (20)

$$\|B_m f - f\|_{L^p(0,B)} \leq \left( \varepsilon + M \sqrt{\delta_m B^\frac{1}{p}} \right) \quad (21)$$

Holds for $x \in (0, B]$ and for sufficiently large $m$. Thus the proof is completed.

**IV. RATE OF CONVERGENCE**

We use Lemma 1 to establish the degree of approximation with (3). Namely, we first approximate $f \in L^p(0,B)$ by $f \in L^p_2((0,B])$ and then use Lemma 1, the J.J. Swetits and definition the K-functional and (4). Also see [2], [4] and [22] for this method.

The following lemma gives upper bound of approximation of $B_m f$ to $f \in L^p(0,B)$ ($m \to \infty$) with help of $\|f\|_{L^p}$ and $\delta_m$. Also it helps the prove of Theorem 3.

**Lemma 1:** Let $f \in L^p_2(0,\infty)$ and $f$ satisfies the condition (13). For all sufficiently large $m$,

$$\|B_m f - f\|_{L^p(0,B)} \leq C_p \left( \|f\|_{L^p_2(0,B)} \right) \delta_m.$$
where $C_0$ is a positive constant and independent of $f$ and $m$.

**Proof:** Now we assume that, $p > 1$ and $x \in (0, B]$. Since $f \in L^p(0, B]$ using Taylor Theorem, we can write that,

$$f(t) - f(x) = f'(x)(t - x) + \int_0^t (t - r)f''(r)dr.$$ 

Applying operator $B_m$ on both side we get

$$B_m(f(t) - f(x); x) = f'(x)B_m(t - x; x) + B_m\left(\int_x^t (t - r)f''(r)dr; x\right) = U_1(x) + U_2(x).$$

Using (23) we get following inequality

$$\|U_1\|_{L^p(0, B]} \leq C_1\left(\|f\|_{L^p(0, B]}\right) \frac{B}{m + 2 B^p}$$

Now we need the Hardy-Littlewood majorante of $f''$ at $x$, which is defined to be

$$\theta_{f''}(x) = \sup_{0 \leq t \leq x \leq \infty} \frac{1}{t - x} \int_x^t |f''(r)|dr. \quad (23)$$

Since $p > 1$ and $f \in L^p$, $\theta_{f''}(x) \in L^p$ according to [29 Theorem 13.5] we get,

$$\int_0^B |\theta_{f''}(x)|^p dx \leq 2\left( \frac{p}{p - 1} \right) \int_0^B \left| f''(x) \right|^p dx. \quad (24)$$

By using (9) and (23) then we obtain on $(0, B]$

$$|U_2(x)| \leq B_m\left(t - x \int_x^t |f''(r)|dr; x\right) \leq \theta_{f''}(x)B_m((t - x)^2; x) \leq \theta_{f''}(x)\delta_m. \quad (25)$$

Then, when we use (24) in above inequality (25)

$$\|U_2\|_{L^p(0, B]} \leq 2^p\left( \frac{p}{p - 1} \right) \|f''\|_{L^p(0, B]}\delta_m.$$ 

Since $\frac{B}{m + 2} \leq \delta_m$ we obtain that,

$$\|U_1\|_{L^p(0, B]} + \|U_2\|_{L^p(0, B]} \leq \left[C_1B^p + 2^p\left( \frac{p}{p - 1} \right)\right] \left(\|f\|_{L^p(0, B]} + \|f''\|_{L^p(0, B]}\right)\delta_m \leq C_p\left(\|f\|_{L^p(0, B]} + \|f''\|_{L^p(0, B]})\right)\delta_m,$$

where $C_p = \left[C_1B^p + 2^p\left( \frac{p}{p - 1} \right)\right]$. Let $p = 1$

$$\int_0^B |f'(x)||B_m(t - x; x)|dx \leq C_2\left(\|f\|_{L^1(0, B]} + \|f''\|_{L^1(0, B]}\right)\frac{B^2}{m + 2} \quad (26)$$

$$\int_0^B |U_2(x)|dx \leq \int_0^B B_m\left(t - x \int_x^t |f''(r)|dr; x\right)dx \leq \|f''\|_{L^1(0, B]} \int_0^B B_m((t - x)^2; x)dx \leq \|f''\|_{L^1(0, B]}B\delta_m \leq B\left(\|f\|_{L^1(0, B]} + \|f''\|_{L^1(0, B]}\right)\delta_m \leq B\left(\|f''\|_{L^1(0, B]}\right)\delta_m \quad (27)$$
Approximation To $L^r$ Interable Functions By Gamma Type Operators

Since $\frac{b}{m+2} \leq \delta_m$ and use (26), (27) in (22), for $p \geq 1$ we have

$$\|U_1\|_{L^p(0,1)} + \|U_2\|_{L^p(0,1)} \leq C_2 \left( \|f\|_{L^p_2(0,1)} \right) B \delta_m$$

where $C_2 = 1 + C_2$.

Thus, the proof of Lemma 1 is completed.

**Theorem 3:** Let $f \in L^p(0, \infty) (1 \leq p \leq \infty)$ and $f$ satisfied the condition (12). For all sufficiently large $m$ and $B > 0$, $B$ is a derivative point off, then the following inequality

$$\|B_m f - f\|_{L^p(0,1)} \leq M_p \left[ \delta_m \|f\|_{L^p_2(0,1)} + \theta_{2,p} (f; \delta_m) \right]$$

(28)

holds. Where $M_p$ is a positive constant, independent of $f$ and $m$.

**Proof:** For all sufficiently large $m$, from Lemma 1 we can write

$$\|B_m h - h\|_{L^p(0,1)} \leq \left\{ \left( \varepsilon + M \delta_m B^{1/p} \right) \|h\|_{L^p(0,1)} ; h \in L^p(0,1) \right\}$$

$$\left( \|h\|_{L_2^p(0,1)} \right) \delta_m ; h \in L_2^p(0,1)$$

where $C_p$ is positive constant which independent of $h,m$ and where $h$ satisfies (12). When $f \in L^p(0, \infty)$ and $g \in L_2^p(0, \infty)$ the condition (12) is satisfies then

$$\|B_m f - f\|_{L^p(0,1)} \leq \|B_m (f - g)\|_{L^p(0,1)} + \|B_m g - g\|_{L^p(0,1)}$$

$$\leq \left( \varepsilon + M \delta_m b^{1/p} \right) \|f - g\|_{L^p(0,1)} + C_p \left( \|f\|_{L_2^p(0,1)} \right) \delta_m$$

$$\leq M_p \left[ \|f - g\|_{L^p(0,1)} + \delta_m \left( \|f\|_{L_2^p(0,1)} \right) \right]$$

where $M_p = \max \left\{ \left( \varepsilon + M \delta_m b^{1/p} \right), C_p \right\}$

Taking infimum over all $g \in L_2^p(0,1)$ which satisfies (12) on the right hand side using the definition of the K-functional we get,

$$\|B_m f - f\|_{L^p(0,1)} \leq M_p \sup_{g \in L_2^p(0,1)} \left[ \|f - g\|_{L^p(0,1)} + \delta_m \left( \|g\|_{L_2^p(0,1)} \right) \right]$$

Since, for a sufficiently large $m$, $\delta_m < 1$ and from (4),

$$K_p(f; \delta_m) \leq \delta_m \|f\|_{L_p(0,1)} + 2 \delta_m \theta_{2,p} (f; \delta_m)$$

$$M_p K_p (f; \delta_m) \leq M_p \left[ \delta_m \|f\|_{L_p(0,1)} + 2 \delta_m \theta_{2,p} (f; \delta_m) \right]$$

We obtain (28),

$$\|B_m f - f\|_{L^p(0,1)} \leq M_p \left[ \delta_m \|f\|_{L_p(0,1)} + \theta_{2,p} (f; \delta_m) \right]$$

Thus the proof of the Theorem 3 is completed.

**REFERENCES**


Approximation To $L^p$ Interable Functions By Gamma Type Operators