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EOQ Inventory Models for Deteriorating Item with Weibull Deterioration and Time-Varying Quadratic Holding Cost

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Abstract

This paper develops an EOQ inventory model for deterioratingitemswith two parameters Weibull deterioration. Shortages are permissible andpartially backlogged. In this model we consider time varying quadratic holding cost and ramp-type demand. The model is developed under two different replenishment policies: (i) Starting with no shortages (ii) Starting with shortages.The aim of this study is to find the optimal solutiontominimizing the total inventory costs for both the above mentioned strategies. To elevate the model a numerical example has been carried out and a sensitivity analysis occurred to study the result of parameters on essential variables and the entire cost of this model.

Keywords: *Inventory, Partial Backlogging, Quadratic Holding Cost, Ramp-Type Demand, Single Item, Shortages, Weibull Deterioration*

I. Introduction

An economic order quantity model determines the quantity a company or a retailer must order to minimize the total inventory cost by balancing the inventory holding cost and fixed ordering cost. In the usual inventory system, it was considered that the buyer pays to vendor as soon as he receives the goods. Goods deteriorate and their valuesdecreases over time. Electronic equipment may become outdated as technology changes; fashion trends depreciate the value of clothes over time; batteries die out as they are old. The outcome of time is even mostimportantfor consumable goods such as foodstuff and drugs.An inventory model for fashion goods deteriorating at the end of the storage period was developed by Whitin [29]. Ghare and Schrader [7] developed an exponentially deteriorating inventory model. A replenishment inventory model for radioactive nuclide generators presented by Emmons [6].An order- level inventory model for decaying items with a deterministic rate of deterioration was developed by Shah and Jaiswal [24]. The retailer receives the delivery of goods and deterioration starts at that moment in all inventory models for decaying items. In real life situation, some customers would like to wait for backlogging during the shortage phase but the others would not. Therefore, the opportunity cost due to lost sales should be considered in the inventory models. Murdeswar[18]; Goyal et al. [11]; Hariga [15];Chakrabarti and Chaudhuri [2]; Hariga and Alyan [16] developed many inventory models assuming that shortages are completely backlogged. Chang and Dye [3] considered that backlogging shortages depends on lead time. An article in the field of deteriorating items with shortages has provided the economic order quantity with a known market demand rate developed by Wee [28]. Sana [22] and Roy et al. [19, 20, 21] developed many inventory models considering partial backlogging rates. The length of the lead time for the next replenishment becomes main factor for determining whether the backlogging will be accepted or not. A detailed review of deteriorating inventory literatures was provided by Goyal and Giri [12]. They considered variable demand rate for many inventory models. Silver and Meal [25] considered time-varying demand rate in their inventory models.

In recent research, Covert and Philip [4] presented an inventory model where the time to deterioration is considered with two parameter Weibull distribution deterioration. Ghosh and Chaudhuri [9] developed an inventory model for two parameters Weibull deteriorating items, with quadratic demand rate and shortages were permissible. Haley and Higgins [13] extended the inventory policy for two part trade credit, where the vendors consider cash discount for paying within a specified period and due in a large credit period. Goyal [10] developed the EOQ model under the conditions of permissible delay in payments. Aggarwal and Jaggi [1] considered deteriorating items to develop ordering policy under the conditions of permissible delay in payments.Sarkar et al. [23] developed an EOQ model for deteriorating items with shortages and time value of money.Teng et al. [27] proposed an EOQ model with deteriorating objects, price perceptive demand, and different selling and purchasing prices under trade credit policy and discussed the result obtained by Goyal [10]. Donaldson [5] provided ananalytical solution procedure of the basic inventory policy for a case of positive linear trend in demand.Giri et al. [8] developed an economic order quantity model for single deteriorating item with ramp type demand and Weibull distribution deterioration. Shortage is allowed and backlogged completely over an infinite planning horizon. Skouri et al. [26] provided an inventory model with time dependent Weibull deterioration rate, partial backlogging of unsatisfied demand and general ramp type demand rate. The model is developed under two different replenishment policies: (a) starting with no shortages and (b) starting with shortages. The backlogging rate is non-increasing function of the waiting time up to the next replenishment. Wu [30] presented an economic order quantity model with ramp type demand and Weibull distribution deterioration. Shortages are allowed partial backlogging and the rate of backlogging is dependent on waiting time for the next replenishment.

The main purpose of this paper is to show that there exist a unique optimal cycle time to minimize the total inventory cost.A numerical illustrationand sensitivity analysisis presented to show the result of the proposed model.

II. Assumptions

The following assumptions are mandatory todevelop mathematical model:

1. The Inventory system consider single item.

2. The inventory level defined by $I(t)$ at time t. i.e.

$$I(t)=\begin{cases} I_a(t)>0, & 0\le t\le t_1 \\ I_b(t)>0, & t_1\le t\le T \end{cases}$$, where $I(t)$ is retailer's stock level.

3. The demand rate $D(t)$ is a ramp type function of time and is defined as:

$$D(t)=\begin{cases} Dt, & t<\mu \\ D\mu, & t\ge\mu \end{cases}$$, Dt is positive and continuous for time $t\in[0,T]$

4. The lead time is zero.

5. Shortagesare allowed and partially backlogged.

6. Cycle horizon is considered as T units of time and replenishment rate is infinite.

7. The rate of deterioration is time dependent, which is two parameters Weibulldistribution deterioration, denotedby $\theta=\alpha\beta t^{\beta-1}$,where $0<\alpha\le 1$, $\beta\ge 1$ and $t>0$ .A value of $t<1$ defines that the failure rate decreases with time. This happens if imperfect items are deterioratingfirst and the failure rate decreases with time. A value of $t=1$ defines that the failure rate is deterministic over time. A value of $t>1$ definesthat the failure rate increases with time.

8. The ordering quantity level is considered as Q .

9. Holding cost $HC(t)$ is a time dependent quadratic function and is defined as $HC(t)=a+bt+ct^2$, where $a>0$, $b>0$ and $c>0$.

III. Notations

The following notations are mandatory to develop mathematical model:

HC : The stock holding cost per year.

S_c : The shortage cost per year.

T : Length of time is considered as annually.

t_1 : Time at which inventory becomes zero.

TC : Total inventory cost.

LS : Lost sale cost.

$I(t)$:Inventory level at time t .

μ : Ramp-type demand function of time.

D_c :Deterioration cost.

IV. Mathematical Modelwithout Shortages:

The inventory model is starting without shortages. At the start of the cycle, the production starts at $t = 0$ and continuesup to $t = t_1$. At this time theinventory level reaches its maximum level and then production is stopped. The inventory lessens to zero due to demand and deteriorationduring $[0, t_1]$ and falls to zero at $t = t_1$. Thus, shortages occur during $[t_1, T]$ which is partially backlogged.Therefore, the inventoryis described by the followingdifferential equations:

$$\frac{dI_a(t)}{dt} = -\theta I_a(t) - D(t), 0 \le t \le t_1 \qquad \ldots(1)$$

with boundary condition $I_a(t_1) = 0$

$$\frac{dI_b(t)}{dt} = -D(t)\delta(T - t), t_1 \le t \le T \qquad \ldots(2)$$

with boundary condition $I_b(t_1) = 0$

Where

$$D(t) = \begin{cases} Dt, & t < \mu \\ D\mu, & t \ge \mu \end{cases}, \qquad and \quad \theta = \alpha\beta t^{\beta-1},$$

There are two cases arise: (i) $t < \mu$ and (ii) $t \ge \mu$

4.1. Case (i) $t < \mu$

From Eq. (1), we have

$$I_a(t) = -De^{-\alpha t^{\beta}}\left[\frac{1}{2}\left(t_1^{2} - t^{2}\right) + \frac{\alpha}{\beta + 2}\left(t_1^{\beta+2} - t^{\beta+2}\right)\right] \qquad \ldots(3)$$

Eq. (2), solved by the following two ways:

$$\frac{dI_b(t)}{dt} = -Dt\delta(T - t), \qquad t_1 \le t \le \mu \qquad (i)$$

$$\frac{dI_b(t)}{dt} = -D\mu\delta(T - t), \qquad \mu \le t \le T,\ \mathrm{I}(\mu_-) = \mathrm{I}(\mu_+) \qquad (ii)$$

We have,

$$I_b(t) = -D\delta\left[\frac{T}{2}\left(t^2 - t_1^{2}\right) - \frac{1}{3}\left(t^3 - t_1^{3}\right)\right] \qquad \ldots(4)$$

and

$$I_b(t) = -D\delta\left[B_1 - \frac{1}{2}Tt_1^2 + \frac{1}{3}t_1^3 - \frac{1}{2}\mu t^2 + Tt\right] \qquad \dots(5)$$

Where

$$B_1 = \frac{1}{2}T\mu^2 + \frac{1}{6}\mu^3 - T\mu$$

Total amount of deterioration during $[0, t_1]$

$$D_c = \int_0^{t_1} D\,t e^{\alpha t^\beta}\,dt - \int_0^{t_1} D\,t\,dt = \frac{\alpha D}{\beta+2}t_1^{\beta+2} \qquad \dots(6)$$

Total cost of holding during $[0, t_1]$ is

$$HC = \int_0^{t_1} H_1(t) I_a(t)\,dt = D\left[\frac{b}{8}t_1^4 + \frac{a}{3}t_1^3 + Bt_1^{\beta+3} - Ct_1^{2\beta+3} + Et_1^{\beta+4} - Ft_1^{2\beta+4}\right] \qquad \dots(7)$$

Where

$$B = \frac{a\alpha}{\beta+3} - \frac{a\alpha^2}{(\beta+1)(\beta+3)}, C = \frac{a\alpha^2}{(\beta+1)(2\beta+3)}, E = \frac{b\alpha}{2(\beta+4)} - \frac{\alpha b}{(\beta+2)(\beta+4)}, F = \frac{b\alpha^2}{(\beta+2)(2\beta+4)}$$

The shortage cost during $[t_1, T]$ is

$$S_c = \int_{t_1}^{T} -I_b(t)\,dt = \int_{t_1}^{\mu} -I_b(t)\,dt + \int_{\mu}^{T} -I_b(t)\,dt = D\delta\left[B_2 - \frac{1}{12}t_1^2\left(6T^2 - 8Tt_1 + 3t_1^2\right)\right] \qquad \dots(8)$$

Where

$$B_2 = \frac{T}{6}\mu^3 - \frac{1}{12}\mu^4 + B_1(T-\mu) - \frac{1}{6}\mu\left(T^3 - \mu^3\right) + \frac{T}{2}\left(T^2 - \mu^2\right)$$

Lost sales cost during $[t_1, T]$ is

$$LS = \int_{t_1}^{\mu}\left(1 - \delta(T-t)\right)D\,t\,dt + D\mu\int_{\mu}^{T}\left(1 - \delta(T-t)\right)dt = \frac{1}{6}D\left[B_3 - 2\delta t_1^3 + 3(\delta T - 1)t_1^2\right] \dots(9)$$

Where

$$B_3 = 6\mu T + 3\delta\mu^2 T - 3\mu\delta T^2 - 3\mu^2 - \delta\mu^3$$

Ordering quantity during $[0, T]$ is

$$OQ = -\int_0^{t_1} e^{\alpha t^\beta} D\,t\,dt - D\delta\int_{t_1}^{\mu} t(T-t)\,dt - D\mu\delta\int_{\mu}^{T}(T-t)\,dt$$

$$OQ = D\left[\frac{\delta}{3}t_1^3 + \left(\frac{1}{2} - \frac{\delta T}{2}\right)t_1^2 + \frac{\alpha}{\beta+2}t_1^{\beta+2} + \frac{1}{2}\mu\delta T^2 - \frac{\delta\mu^2}{2}T + \frac{1}{6}\delta\mu^3\right] \qquad \dots(10)$$

Total cost during $[0, T]$ is the sum of deterioration cost, holding cost, shortage cost and lost sales cost is given by

$$TC_1(t_1) = HC + S_c + LS + D_c$$

$$TC_1(t_1) = D\left[\begin{array}{l}\frac{\alpha}{\beta+2}t_1^{\beta+2} + \left(\frac{b}{8} - \frac{\delta}{4}\right)t_1^4 + \frac{1}{3}(a + 2\delta T - \delta)t_1^3 + Bt_1^{\beta+3} - Ct_1^{2\beta+3} \\ + Et_1^{\beta+4} - Ft_1^{2\beta+4} + \frac{1}{2}\left(\delta T - \delta T^2 - 1\right)t_1^2 + \frac{B_3}{6} + \delta B_2\end{array}\right] \qquad \dots(11)$$

4.1.1. Solution:

$$\frac{\partial TC_1}{\partial t_1} = D\begin{bmatrix} \alpha t_1^{\beta+1} + \left(\frac{b}{2} - \delta\right)t_1^3 + (a + 2\delta T - \delta)t_1^2 + B(\beta+3)t_1^{\beta+2} - C(2\beta+3) \\ t_1^{2\beta+2} + E(\beta+4)t_1^{\beta+3} - F(2\beta+4)t_1^{2\beta+3} + \left(\delta T - \delta T^2 - 1\right)t_1 \end{bmatrix} \quad \ldots(12)$$

$$\frac{\partial^2 TC_1}{\partial t_1^2} = D\begin{bmatrix} \alpha(\beta+1)t_1^{\beta} + 3\left(\frac{b}{2} - \delta\right)t_1^2 + 2(a + 2\delta T - \delta)t_1 + B(\beta+2) \\ (\beta+3)t_1^{\beta+1} - C(2\beta+2)(2\beta+3)t_1^{2\beta+1} + E(\beta+3)(\beta+4)t_1^{\beta+2} \\ -F(2\beta+3)(2\beta+4)t_1^{2\beta+2} + \delta T - \delta T^2 - 1 \end{bmatrix} \quad \ldots(13)$$

Main objective is to minimize the total relevant cost for the inventory model starting without shortages. The essential condition to minimize the total relevant cost is $\frac{\partial TC_1}{\partial t_1} = 0$, we have

$$\alpha t_1^{\beta+1} + \left(\frac{b}{2} - \delta\right)t_1^3 + (a + 2\delta T - \delta)t_1^2 + B(\beta+3)t_1^{\beta+2} - C(2\beta+3)$$

$$t_1^{2\beta+2} + E(\beta+4)t_1^{\beta+3} - F(2\beta+4)t_1^{2\beta+3} + \left(\delta T - \delta T^2 - 1\right)t_1 = 0 \quad \ldots(14)$$

Using the software Mathematica, we can calculate the optimal value of t_1 by Eq. (14) and the optimal value $TC_1(t_1)$ of the total relevant cost is determined by Eq. (11). The optimal value of t_1 satisfy the sufficient condition for minimizing total relevant cost $TC_1(t_1)$ is

$$\frac{\partial^2 TC_1}{\partial t_1^2} > 0 \quad \ldots(15)$$

The sufficient condition is satisfied.

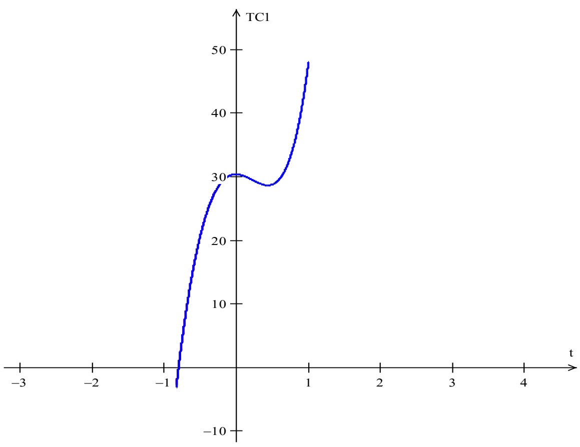

Fig.1. Graphical representation of TC_1 and t_1.

4.1.2. Numerical example:

Let us consider

$a = \$1/unit, b = \$0.5/unit, S_c = \$0.5/unit, D_c = \$5/unit, \alpha = 0.1, \beta = 2, T = 1year, D = 100,$

$\delta = 1, \mu = 0.5\ year$

$B_1 = -0.35, B_2 = 0.14, B_3 = 1.4, B = 0.02, C = 0.0005, E = 0.002, F = 0.00015$

Thus, the optimal value of t_1 is $t_1^* \rightarrow 0.44 < \mu$. The optimal ordering quantity is $OQ^* = 17.5$. The minimum relevant cost is $TC_1^* = 28.6$.

4.1.3. SensitivityAnalysis:

To know, how the optimal solution is affected by the values of constraints, we derive the sensitivity analysis for some constraints. The specific values of some constraints are increased or decreased by $+25\%$, -25% and $+50\%$, -50% .Thus,we computethe values of t_1^* and TC_1^* with the help of increased or decreased values of S_c and D_c .The result of the minimum relevant cost is existing in the following table 1.

Table: 1

Parameters	Actual Values	$+50\%$ Increased	-50% Decreased	$+25\%$ Increased	-25% Decreased
S_c	0.5	0.75	0.25	0.625	0.375
D_c	5	7.5	2.5	6.25	3.75
t_1^*	0.44	0.5	0.23	0.5	0.34
TC_1^*	28.6	40.75	14.93	34.7	21.98
OQ^*	17.5	18.9	14.99	18.9	15.93

From the result of above table, we observe that total relevant cost and ordering quantity is much affected by deterioration cost and shortage cost, and other parameters are less sensitive.

4.1.4. When Holding Cost is a Quadratic Function:

We have

$$HC = \int_0^{t_1} H_1(t) I_a(t)\,dt = \int_0^{t_1} \left(a + bt + ct^2\right) D e^{-\alpha t^{\beta}} \left[\frac{1}{2}\left(t_1^2 - t^2\right) + \frac{\alpha}{\beta+2}\left(t_1^{\beta+2} - t^{\beta+2}\right)\right] dt$$

$$HC = D\left[\frac{c}{15}t_1^5 + \frac{b}{8}t_1^4 + \frac{a}{3}t_1^3 + Bt_1^{\beta+3} - Ct_1^{2\beta+3} + Et_1^{\beta+4} - Ft_1^{2\beta+4} + F_1 t_1^{5+\beta} - F_2 t_1^{2\beta+5}\right] \quad \ldots(16)$$

Where

$$B = \frac{a\alpha}{\beta+3}\left(1 - \frac{1}{(\beta+1)}\right), C = \frac{a\alpha^2}{(\beta+1)(2\beta+3)}, E = \frac{b\alpha}{(\beta+4)}\left(\frac{1}{2} - \frac{1}{(\beta+2)}\right),$$

$$F = \frac{b\alpha^2}{(\beta+2)(2\beta+4)}, F_1 = \frac{\alpha c}{(\beta+5)}\left(\frac{1}{3} - \frac{1}{(\beta+3)}\right), F_2 = \frac{c\alpha^2}{(\beta+3)(2\beta+5)}$$

Total cost is the sum of quadratic holding cost, shortage cost, deterioration cost and lost sales cost during $[0, T]$

$$TC_1(t_1) = HC + S_c + LS + D_c$$

$$TC_1(t_1) = D\begin{bmatrix} \frac{\alpha}{\beta+2}t_1^{\beta+2} + \frac{c}{15}t_1^5 + \left(\frac{b}{8} - \frac{\delta}{4}\right)t_1^4 + \frac{1}{3}(a + 2\delta T - \delta)t_1^3 + Bt_1^{\beta+3} - Ct_1^{2\beta+3} \\ + Et_1^{\beta+4} - Ft_1^{2\beta+4} + F_1 t_1^{5+\beta} - F_2 t_1^{2\beta+5} + \frac{1}{2}(\delta T - 1 - \delta T^2)t_1^2 + \delta B_2 + \frac{1}{6}B_3 \end{bmatrix} \quad \ldots(17)$$

4.1.5. Solution:

$$\frac{\partial TC_1}{\partial t_1} = D\begin{bmatrix} \alpha t_1^{\beta+1} + \frac{c}{3}t_1^4 + \left(\frac{b}{2} - \delta\right)t_1^3 + (a + 2\delta T - \delta)t_1^2 + B(\beta+3)t_1^{\beta+2} - C(2\beta+3)t_1^{2\beta+2} \\ + E(\beta+4)t_1^{\beta+3} - F(2\beta+4)t_1^{2\beta+3} + F_1(\beta+5)t_1^{\beta+4} - F_2(2\beta+5)t_1^{2\beta+4} + \left(\delta T - 1 - \delta T^2\right)t_1 \end{bmatrix} \ldots(18)$$

$$\frac{\partial^2 TC_1}{\partial t_1^2} = D\left[\begin{array}{l} \alpha(\beta+1)t_1^{\beta} + \frac{4c}{3}t_1^3 + 3\left(\frac{b}{2}-\delta\right)t_1^2 + 2(a+2\delta T-\delta)t_1 + B(\beta+2)(\beta+3)t_1^{\beta+1} \\ -C(2\beta+2)(2\beta+3)t_1^{2\beta+1} + E(\beta+3)(\beta+4)t_1^{\beta+2} - F(2\beta+3)(2\beta+4)t_1^{2\beta+2} \\ +F_1(\beta+4)(\beta+5)t_1^{\beta+3} - F_2(2\beta+4)(2\beta+5)t_1^{2\beta+3} + \delta T - 1 - \delta T^2 \end{array}\right] \quad \ldots(19)$$

Main objective is to minimize the total relevant cost of the inventory model starting without shortages. The essential condition to minimize the total relevant cost is $\frac{\partial TC_1}{\partial t_1} = 0$, we have

$$\alpha t_1^{\beta+1} + \frac{c}{3}t_1^4 + \left(\frac{b}{2}-\delta\right)t_1^3 + (a+2\delta T-\delta)t_1^2 + B(\beta+3)t_1^{\beta+2} - C(2\beta+3)t_1^{2\beta+2} + E(\beta+4)$$
$$t_1^{\beta+3} - F(2\beta+4)t_1^{2\beta+3} + F_1(\beta+5)t_1^{\beta+4} - F_2(2\beta+5)t_1^{2\beta+4} + (\delta T - 1 - \delta T^2)t_1 = 0 \quad \ldots(20)$$

Using the software Mathematica, we can calculate the optimal value of t_1 by Eq. (20) and the optimal value $TC_1(t_1)$ of the total relevant cost is determined by Eq. (17). The optimal value of t_1 satisfy the sufficient condition for minimizing total relevant cost $TC_1(t_1)$ is

$$\frac{\partial^2 TC_1}{\partial t_1^2} > 0 \quad \ldots(21)$$

The sufficient condition is satisfied.

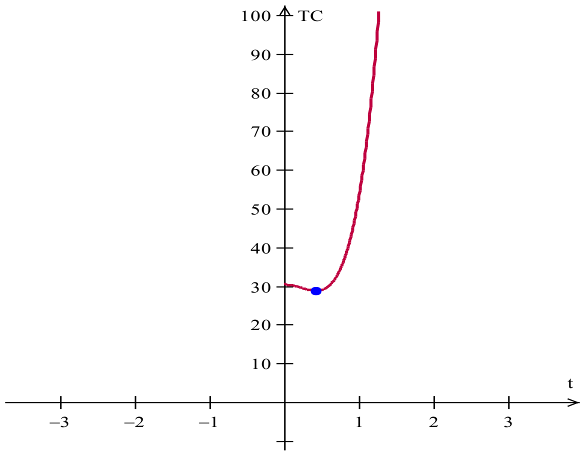

Fig.2. Graphical representation of TC_1 and t_1.

4.1.6. Numerical Example:

Let us consider

$a = \$1/unit, b = \$0.5/unit, c = 1, S_c = \$0.5/unit, D_c = \$5/unit, \alpha = 0.1, \beta = 2, T = 1 year, D = 100,$

$\delta = 1, \mu = 0.5\ year$

$B_1 = -0.35, B_2 = 0.14, B_3 = 1.4, B = 0.01, C = 0.0005, E = 0.002, F = 0.00015, F_1 = 0.002, F_2 = 0.0002$

Thus, the optimal value of t_1 is $t_1^* \to 0.425 < \mu$. The optimum ordering quantity is $OQ^* = 17.5$. The minimum relevant cost is $TC_1^* = 28.69$.

4.1.7. SensitivityAnalysis:

To know, how the optimal solution is affected by the values of constraints, we derive the sensitivity analysis for some constraints. The specific values of some constraints are increased or decreased by $+25\%$, -25% and $+50\%$, -50% .Thus,we computethe values of t_1^* and TC_1^* with the help of increased or decreased values of S_c and D_c .The result of minimum relevant cost is existing in the following table 2.

Table: 2

Parameters	Actual Values	+50% Increased	−50% Decreased	+25% Increased	−25% Decreased
S_c	0.5	0.75	0.25	0.625	0.375
D_c	5	7.5	2.5	6.25	3.75
t_1^*	0.425	0.58	0.23	0.51	0.33
TC_1^*	28.7	40.61	14.93	34.9	22.01
OQ^*	17.5	18.9	14.99	18.9	15.93

From the result of above table, we observe that total relevant cost and ordering quantity is much affected by deterioration cost and shortage cost, and other parameters are less sensitive.

4.2. Case (ii): $t \geq \mu$

The differential Eq.(1) becomes

$$\frac{dI_a(t)}{dt} + \theta I_a(t) = -Dt, \qquad 0 \leq t \leq \mu, \ \mathrm{I}(\mu_-) = \mathrm{I}(\mu_+)$$

$$\frac{dI_a(t)}{dt} + \theta I_a(t) = -D\mu, \qquad \mu \leq t \leq t_1, \quad \mathrm{I}(t_1) = 0$$

We have

$$I_a(t) = De^{-\alpha t^\beta}\left[C_1 - \frac{1}{2}t^2 - \frac{\alpha}{\beta+2}t^{\beta+2} + \mu t_1 + \frac{\alpha\mu}{\beta+1}t_1^{\beta+1}\right], \quad 0 \leq t \leq \mu \qquad \ldots(22)$$

Where

$$C_1 = \left(\frac{1}{\beta+2} - \frac{1}{\beta+1}\right)\alpha\mu^{\beta+2} - \frac{1}{2}\mu^2$$

and $I_a(t) = D\mu e^{-\alpha t^\beta}\left[t_1 + \frac{\alpha}{\beta+1}t_1^{\beta+1} - t - \frac{\alpha}{\beta+1}t^{\beta+1}\right], \quad \mu \leq t \leq t_1 \qquad \ldots(23)$

The differential Eq. (2) becomes

$$\frac{dI_b(t)}{dt} = -D\mu\delta(T-t), \qquad t_1 \leq t \leq T, \ I_b(t_1) = 0$$

We have $I_b(t) = D\mu\delta\int_{t_1}^{t}(T-t)\,dt = D\mu\delta\left[T(t-t_1) - \frac{1}{2}\left(t^2 - t_1^2\right)\right], \quad t_1 \leq t \leq T \qquad \ldots(24)$

Total amount of deterioration during $[0, t_1]$

$$D_c = \int_0^\mu Dte^{\alpha t^\beta}dt + D\mu\int_\mu^{t_1} e^{\alpha t^\beta}dt - \int_\mu^{t_1} D\mu\,dt = D\left[S + \frac{\alpha\mu}{\beta+1}t_1^{\beta+1}\right] \qquad \ldots(25)$$

Where,

$$S = \frac{\mu^2}{2} + \left(\frac{1}{\beta+2} - \frac{1}{\beta+1}\right)\alpha\mu^{\beta+2}$$

Total holding cost during $[0, t_1]$ is

$$HC = \int_0^{t_1} H_1(t)I_a(t)\,dt = \int_0^\mu (a+bt)I_a(t)\,dt + \int_\mu^{t_1}(a+bt)I_a(t)\,dt$$

$$HC = D\left[\frac{b\mu}{6}t_1^{\,3} + \frac{a\mu}{2}t_1^{\,2} + \left(C_3 + C_5\mu\right)t_1 + \left(C_4 + \mu C_8\right)t_1^{\,\beta+1} + C_7\mu t_1^{\,\beta+2} + C_6\mu t_1^{\,\beta+3} + C_9\mu t_1^{\,2\beta+2} + C_{10}\mu t_1^{\,2\beta+3} + C_2\right] \quad \ldots(26)$$

Where,

$$C_2 = \left(\frac{1}{2(\beta+2)^2} - \frac{1}{(\beta+1)(2\beta+3)}\right)b\alpha^2\mu^{2\beta+4} + \left(\frac{1}{(\beta+2)(2\beta+3)} - \frac{1}{2(\beta+1)^2}\right)a\alpha^2\mu^{2\beta+3} - \frac{b\alpha C_1}{\beta+2}\mu^{\beta+2} - \frac{a\alpha C_1}{\beta+1}\mu^{\beta+1} + \frac{5}{24}b\mu^4 + \frac{1}{3}a\mu^3$$

$$+\left(\frac{1}{2(\beta+4)} - \frac{1}{(\beta+2)(\beta+4)} + \frac{1}{(\beta+1)(\beta+3)} - \frac{1}{(\beta+3)}\right)b\alpha\mu^{\beta+4} + \left(\frac{1}{(\beta+1)(\beta+2)} - \frac{1}{(\beta+2)} - \frac{1}{2(\beta+3)}\right)a\alpha\mu^{\beta+3} + \frac{bC_1}{2}\mu^2 + a\mu C_1,$$

$$C_3 = a\mu^2 - \frac{a\alpha}{\beta+1}\mu^{\beta+2} + \frac{b\mu^3}{2} - \frac{b\alpha\mu^{\beta+3}}{\beta+2}, C_4 = \frac{\alpha^2\mu^2}{\beta+1} - \frac{a\mu\alpha^2}{(\beta+1)^2}\mu^{\beta+1} + \frac{b\alpha\mu^3}{2(\beta+1)} - \frac{b\alpha^2\mu^{\beta+3}}{(\beta+1)(\beta+2)}$$

$$C_5 = \frac{a\alpha\mu^{\beta+1}}{(\beta+1)} - a\mu - \frac{b\mu^2}{2} + \frac{b\alpha\mu^{\beta+2}}{(\beta+2)}, C_6 = \frac{b\alpha}{2(\beta+1)} - \frac{b\alpha}{(\beta+1)(\beta+3)} - \frac{b\alpha}{(\beta+2)} + \frac{b\alpha}{(\beta+3)},$$

$$C_7 = \left(1 - \frac{1}{(\beta+1)}\right)\frac{a\alpha}{(\beta+2)}, C_9 = \left(\frac{1}{(2\beta+2)} - \frac{1}{(\beta+1)}\right)\frac{a\alpha^2}{(\beta+1)},$$

$$C_8 = \frac{a\alpha^2\mu^{\beta+1}}{(\beta+1)^2} - \frac{a\alpha\mu}{\beta+1} - \frac{b\alpha\mu^2}{2(\beta+1)} + \frac{b\mu^{\beta+2}\alpha^2}{(\beta+2)(\beta+1)}, C_{10} = \left(\frac{1}{(2\beta+3)} - \frac{1}{(\beta+2)}\right)\frac{b\alpha^2}{(\beta+1)}$$

The shortage cost during $[t_1, T]$ is

$$S_c = \int_{t_1}^{T} -I_b(t)\,dt = D\mu\delta\int_{t_1}^{T}\left[T(t - t_1) - \frac{1}{2}\left(t^2 - t_1^{\,2}\right)\right]dt = D\mu\delta\left[\frac{T^3}{3} - T^2 t_1 + Tt_1^2 - \frac{t_1^3}{3}\right] \quad \ldots(27)$$

Lost sales cost during $[t_1, T]$ is

$$LS = D\mu\int_{t_1}^{T}\left(1 - \delta(T - t)\right)dt = -\frac{1}{2}D\mu(T - t_1)(-2 + T\delta - \delta t_1) \quad \ldots(28)$$

Ordering quantity is during $[0, T]$ is

$$OQ = \int_0^{\mu} D\,te^{\alpha t^{\beta}}\,dt + D\mu\int_{\mu}^{t_1} dt + D\mu\delta\int_{t_1}^{T}(T - t)\,dt$$

$$= D\left[\frac{\mu\delta}{2}t_1^{\,2} + (1 - \delta T)\mu t_1 - \frac{\mu^2}{2} + \frac{\alpha}{\beta+2}\mu^{\beta+2} + \frac{\mu\delta T^2}{2}\right] \quad \ldots(29)$$

Total cost is the sum of deterioration cost, holding cost, shortage cost and lost sales cost during $[0, T]$

$$TC_2(t_1) = HC + S_c + LS + D_c$$

$$TC_2(t_1) = D\left[\begin{array}{l} C_{10}\mu t_1^{\,2\beta+3} + C_9\mu t_1^{\,2\beta+2} + C_6\mu t_1^{\,\beta+3} + C_7\mu t_1^{\,\beta+2} + \left(C_4 + \mu C_8 + \dfrac{\alpha\mu}{\beta+1}\right) \\ t_1^{\,\beta+1} + \mu\left(\dfrac{b}{6} - \dfrac{\delta}{3}\right)t_1^3 + \left(\dfrac{a\mu}{2} - \dfrac{\mu\delta}{2} + \mu\delta T\right)t_1^{\,2} + \left(\begin{array}{l} C_3 + C_5\mu - \mu\delta T^2 \\ -\mu(1 - \delta T)\end{array}\right)t_1 \\ + \dfrac{\mu\delta T^3}{3} - \dfrac{\mu}{2}\delta T^2 - \mu T + C_2 + S \end{array}\right] \ldots(30)$$

4.2.1.Solution:

$$\frac{\partial TC_2}{\partial t_1} = D\left[\begin{array}{l} C_{10}\mu(2\beta+3)t_1^{2\beta+2} + C_9\mu(2\beta+2)t_1^{2\beta+1} + C_6\mu(\beta+3)t_1^{\beta+2} \\ +C_7\mu(\beta+2)t_1^{\beta+1} + \left(C_4+\mu C_8+\dfrac{\alpha\mu}{\beta+1}\right)(\beta+1)t_1^{\beta} + 3\mu\left(\dfrac{b}{6}-\dfrac{\delta}{3}\right)t_1^2 \\ +2\left(\dfrac{a\mu}{2}-\dfrac{\mu\delta}{2}+\mu\delta T\right)t_1 + C_3 + C_5\mu - \mu\delta T^2 - \mu(1-\delta T) \end{array}\right] \quad \ldots(31)$$

$$\frac{\partial^2 TC_2}{\partial t_1^2} = D\left[\begin{array}{l} C_{10}\mu(2\beta+2)(2\beta+3)t_1^{2\beta+1} + C_9\mu(2\beta+1)(2\beta+2)t_1^{2\beta} + C_6\mu \\ (\beta+2)(\beta+3)t_1^{\beta+1} + C_7\mu(\beta+1)(\beta+2)t_1^{\beta} + \left(C_4+\mu C_8+\dfrac{\alpha\mu}{\beta+1}\right) \\ \beta(\beta+1)t_1^{\beta-1} + \mu(b-2\delta)t_1 + a\mu - \mu\delta + 2\mu\delta T \end{array}\right] \quad \ldots(32)$$

Main objective is to minimize the total relevant cost of the inventory model starting without shortages. The essential condition to minimize the total relevant cost is $\frac{\partial TC_2}{\partial t_1} = 0$, we have

$$C_{10}\mu(2\beta+3)t_1^{2\beta+2} + C_9\mu(2\beta+2)t_1^{2\beta+1} + C_6\mu(\beta+3)t_1^{\beta+2} + C_7\mu(\beta+2)t_1^{\beta+1}$$
$$+\left(C_4+\mu C_8+\frac{\alpha\mu}{\beta+1}\right)(\beta+1)t_1^{\beta} + 3\mu\left(\frac{b}{6}-\frac{\delta}{3}\right)t_1^2 + 2\left(\frac{a\mu}{2}-\frac{\mu\delta}{2}+\mu\delta T\right)t_1 + C_3$$
$$+C_5\mu - \mu\delta T^2 - \mu(1-\delta T) = 0 \quad \ldots(33)$$

Using the software Mathematica, we can calculate the optimal value of t_1 by Eq. (33) and the optimal value $TC_2(t_1)$ of the total relevant cost is determined by Eq. (30). The optimal value of t_1 satisfy the sufficient condition for minimizing total relevant cost $TC_2(t_1)$ is

$$\frac{\partial^2 TC_2}{\partial t_1^2} > 0 \quad \ldots(34)$$

The sufficient condition is satisfied.

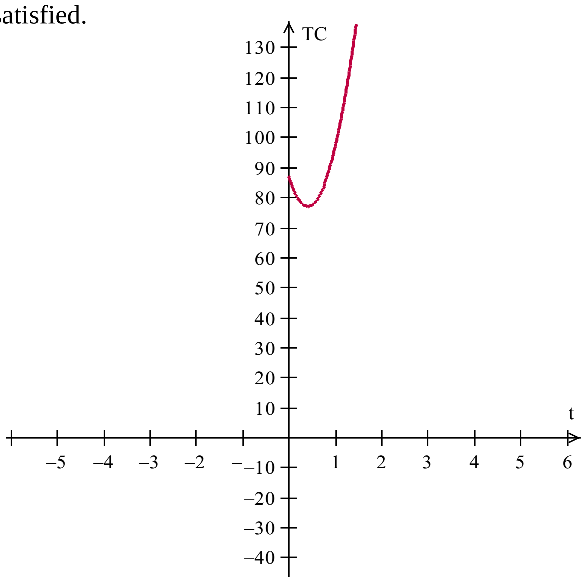

Fig.3. Graphical representation of TC_2 and t_1.

4.2.2. Numerical example:
Let us consider

$a = \$1/unit, b = \$0.5/unit, S_c = \$0.5/unit, D_c = \$5/unit, \alpha = 0.1, \beta = 2, T = 1\,year,$

$D = 100, \delta = 1, \mu = 0.5\ year$

$C_1 = -0.12, S = 0.12,\ C_2 = -0.02, C_3 = 0.28, C_4 = 0.002, C_5 = -0.56, C_6 = 0.0025,$

$C_7 = 0.017, C_8 = -0.018, C_9 = -0.0005, C_{10} = -0.0002$

Thus, the optimal value of t_1 is $t_1^* \to 0.54 > \mu$. The optimum ordering quantity is $OQ^* = 19.95$. The minimum relevant cost is $TC_2^* = 77.03$.

4.2.3. When Holding Cost is a Quadratic Function:
Total Quadratic holding cost during $[0, t_1]$ is

$$HC = \int_0^{t_1} H_1(t) I_a(t)\,dt = \int_0^{\mu} \left(a + bt + ct^2\right) I_a(t)\,dt + \int_{\mu}^{t_1} \left(a + bt + ct^2\right) I_a(t)\,dt$$

$$HC = D\begin{bmatrix} \frac{c\mu}{12}t_1^4 + \frac{b\mu}{6}t_1^3 + \frac{a\mu}{2}t_1^2 + \mu\left(C_3 - C_4\right)t_1 + \mu\left(\frac{C_3\alpha}{\beta+1} - C_7\right)t_1^{\beta+1} + C_6\mu t_1^{\beta+2} \\ + C_5\mu t_1^{\beta+3} - C_{10}\mu t_1^{\beta+4} + C_8\mu t_1^{2\beta+2} + C_9\mu t_1^{2\beta+3} + C_{11}\mu t_1^{2\beta+4} + C_2 \end{bmatrix} \quad \ldots(35)$$

Where,

$$C_2 = a\mu C_1 - \frac{a}{6}\mu^3 - \frac{a\alpha}{(\beta+2)(\beta+3)}\mu^{\beta+3} - \frac{a\alpha}{\beta+1}\mu^{\beta+1}C_1 + \frac{a\alpha}{2(\beta+3)}\mu^{\beta+3} + \frac{a\alpha^2}{(\beta+2)(2\beta+3)}\mu^{2\beta+3} + \frac{b\mu^2}{2}C_1 - \frac{b\mu^4}{8} - \frac{b\alpha\mu^{\beta+4}}{(\beta+2)(\beta+4)}$$
$$- \frac{b\alpha\mu^{\beta+2}}{\beta+2}C_1 + \frac{b\alpha}{2(\beta+4)}\mu^{\beta+4} + \frac{b\alpha^2\mu^{2\beta+4}}{(\beta+2)(2\beta+4)} + \frac{\mu^3}{3}cC_1 - \frac{c}{10}\mu^5 - \frac{c\alpha}{(\beta+3)}\mu^{\beta+3}C_1 - \frac{c\alpha^2\mu^{2\beta+5}}{(\beta+1)(2\beta+4)}$$
$$+ \frac{c\alpha^2}{(\beta+2)(2\beta+5)}\mu^{2\beta+5} + \frac{a}{2}\mu^3 + \frac{a\alpha\mu^{\beta+3}}{(\beta+1)(\beta+2)} - \frac{a\alpha\mu^{\beta+3}}{(\beta+2)} - \frac{a\alpha^2\mu^{2\beta+3}}{(\beta+1)(2\beta+2)} + \frac{b}{3}\mu^4 + \frac{b\alpha\mu^{\beta+4}}{(\beta+1)(\beta+3)} - \frac{b\alpha\mu^{\beta+4}}{(\beta+3)} - \frac{b\alpha^2\mu^{2\beta+4}}{(\beta+1)(2\beta+3)}$$
$$+ \frac{c}{4}\mu^5 + \left(\frac{1}{(\beta+1)} + 1\right)\frac{c\alpha\mu^{\beta+5}}{(\beta+4)} - \frac{\alpha}{(\beta+2)(\beta+5)}\mu^{\beta+5} - \frac{c\alpha}{2(\beta+5)}\mu^{\beta+5}, C_3 = \left(a\mu - \frac{a\alpha\mu^{\beta+1}}{\beta+1} + \frac{b\mu^2}{2} - \frac{b\alpha\mu^{\beta+2}}{\beta+2} + \frac{c\mu^3}{3} - \frac{c\alpha\mu^{\beta+3}}{\beta+3}\right),$$

$$C_4 = a\mu - \frac{a\alpha\mu^{\beta+1}}{(\beta+1)} - \frac{b\mu^2}{2} - \frac{c\mu^3}{3} + \frac{b\alpha\mu^{\beta+2}}{(\beta+2)} + \frac{c\alpha\mu^{\beta+3}}{(\beta+3)}, C_5 = \frac{b\alpha}{2(\beta+1)} - \frac{b\alpha}{(\beta+1)(\beta+3)} - \frac{b\alpha}{(\beta+2)} + \frac{b\alpha}{(\beta+3)}, C_6 = \left(1 - \frac{1}{(\beta+1)}\right)\frac{a\alpha}{(\beta+2)},$$

$$C_7 = \alpha\mu\left(\frac{6a - 6a\alpha\mu^{\beta} - 3b\mu - 2c\mu^2}{6(\beta+1)}\right) + \frac{b\alpha^2\mu^{\beta+2}}{(\beta+2)(\beta+1)} + \frac{c\alpha^2\mu^{\beta+3}}{(\beta+1)(\beta+3)}, C_8 = -\frac{a\alpha^2}{2(\beta+1)^2},$$

$$C_9 = \left(\frac{1}{(2\beta+3)} - \frac{1}{(\beta+2)}\right)\frac{b\alpha^2}{(\beta+1)}, C_{10} = \frac{c\alpha}{(\beta+1)(\beta+4)} + \frac{c\alpha}{(\beta+4)} + \frac{c\alpha}{(\beta+3)} + \frac{c\alpha}{3(\beta+1)},$$

$$C_{11} = \left(\frac{1}{(2\beta+4)} - \frac{1}{(\beta+3)}\right)\frac{c\alpha^2}{(\beta+1)}$$

Total cost is the sum of quadratic holding cost, shortage cost, deterioration cost and lost sales cost during $[0, T]$

$$TC_2(t_1) = HC + S_c + LS + D_c$$

$$TC_2(t_1) = D\begin{bmatrix} \frac{c\mu}{12}t_1^4 + \mu\left(\frac{b}{6} - \frac{\delta}{3}\right)t_1^3 + \mu\left(\delta T + \frac{a}{2} - \frac{\delta}{2}\right)t_1^2 + \mu\left(C_3 - C_4 - \delta T^2 + \delta T - 1\right)t_1 + \mu\left(\frac{\alpha}{\beta+1} + \frac{C_3\alpha}{\beta+1} - C_7\right)t_1^{\beta+1} \\ + C_6\mu t_1^{\beta+2} + C_5\mu t_1^{\beta+3} - C_{10}\mu t_1^{\beta+4} + C_8\mu t_1^{2\beta+2} + C_9\mu t_1^{2\beta+3} + C_{11}\mu t_1^{2\beta+4} + \frac{\mu\delta T^3}{3} - \frac{1}{2}\mu\delta T^2 C_2 + \mu T + S \end{bmatrix} \quad \ldots(36)$$

4.2.4. Solution:

$$\frac{\partial TC_2}{\partial t_1} = D\begin{bmatrix} \frac{c\mu}{3}t_1^3 + \mu\left(\frac{b}{2}-\delta\right)t_1^2 + \mu\left(2\delta T + a - \delta\right)t_1 + \mu\left(C_3 - C_4 - \delta T^2 + \delta T - 1\right) + \mu\left(\alpha + C_3\alpha - C_7\left(\beta+1\right)\right)t_1^{\beta} \\ +C_6\mu\left(\beta+2\right)t_1^{\beta+1} + C_5\mu\left(\beta+3\right)t_1^{\beta+2} - C_{10}\mu\left(\beta+4\right)t_1^{\beta+3} + C_8\mu\left(2\beta+2\right)t_1^{2\beta+1} + C_9\mu\left(2\beta+3\right)t_1^{2\beta+2} \\ +C_{11}\mu\left(2\beta+4\right)t_1^{2\beta+3} \end{bmatrix} \quad \ldots(37)$$

$$\frac{\partial^2 TC_2}{\partial t_1^2} = D\begin{bmatrix} c\mu t_1^2 + \mu\left(b-2\delta\right)t_1 + \mu\left(2\delta T + a - \delta\right) + \mu\beta\left(\alpha + C_3\alpha - C_7\left(\beta+1\right)\right)t_1^{\beta-1} \\ +C_6\mu\left(\beta+1\right)\left(\beta+2\right)t_1^{\beta} + C_5\mu\left(\beta+2\right)\left(\beta+3\right)t_1^{\beta+1} - C_{10}\mu\left(\beta+3\right)\left(\beta+4\right)t_1^{\beta+2} \\ +C_8\mu\left(2\beta+1\right)\left(2\beta+2\right)t_1^{2\beta} + C_9\mu\left(2\beta+2\right)\left(2\beta+3\right)t_1^{2\beta+1} + C_{11}\mu\left(2\beta+3\right)\left(2\beta+4\right)t_1^{2\beta+2} \end{bmatrix} \quad \ldots(38)$$

Main objective is to minimize the total relevant cost for the inventory model starting without shortages. The essential condition to minimize the total relevant cost is $\frac{\partial TC_2}{\partial t_1} = 0$, we have

$$\frac{c\mu}{3}t_1^3 + \mu\left(\frac{b}{2}-\delta\right)t_1^2 + \mu\left(2\delta T + a - \delta\right)t_1 + \mu\left(C_3 - C_4 - \delta T^2 + \delta T - 1\right) + \mu\left(\alpha + C_3\alpha - C_7\left(\beta+1\right)\right)t_1^{\beta}$$
$$+C_6\mu\left(\beta+2\right)t_1^{\beta+1} + C_5\mu\left(\beta+3\right)t_1^{\beta+2} - C_{10}\mu\left(\beta+4\right)t_1^{\beta+3} + C_8\mu\left(2\beta+2\right)t_1^{2\beta+1} + C_9\mu\left(2\beta+3\right)t_1^{2\beta+2}$$
$$+C_{11}\mu\left(2\beta+4\right)t_1^{2\beta+3} = 0 \quad \ldots(39)$$

Using the software Mathematica, we can calculate the optimal value of t_1 by Eq. (39) and the optimal value $TC_2\left(t_1\right)$ of the total relevant cost is determined by Eq. (36). The optimal value of t_1 satisfy the sufficient condition for minimizing total relevant cost $TC_2\left(t_1\right)$ is

$$\frac{\partial^2 TC_2}{\partial t_1^2} > 0 \quad \ldots(40)$$

The sufficient condition is satisfied.

4.2.5. Numerical example:
Let us consider

$a = \$1/unit, b = \$0.5/unit, c = 1, S_c = \$0.5/unit, D_c = \$5/unit, \alpha = 0.1, \beta = 2, T = 1 year, D = 100, \delta = 1,$
$\mu = 0.5\ year$

$C_1 = -0.12, C_2 = -0.02, C_3 = 0.6, C_4 = 0.39, C_5 = 0.0025, C_6 = 0.017, C_7 = 0.013, C_8 = -0.0005,$
$C_9 = -0.0002, C_{10} = 0.053, C_{11} = -0.00025$

Thus, the optimal value of t_1 is $t_1^* \rightarrow 0.5 = \mu$. The optimum ordering quantity is $OQ^* = 19.95$. The minimum relevant cost is $TC_2^* = 82.8$.

V. Mathematical Model with Shortages:

The cycle starts with shortages that occur during the period $\left[0, t_1\right]$ and shortages are partially backlogged. Replenishment grasps the inventory level up to Q after time t_1. The inventory level depletes and falls to zero at $t = T$ because of demand and deterioration during $\left[t_1, T\right]$. Two cases occur

(i) $t_1 < \mu$

(ii) $t_1 \geq \mu$

5.1. Case (i) $t_1 < \mu$

Therefore, the inventory $I(t)$ is described by the system of differential equations during $[0, T]$

$$\frac{dq_1(t)}{dt} = -D(t)\,\delta(t_1 - t), \quad 0 \le t \le t_1, \quad q(0) = 0 \qquad \ldots(41)$$

$$\frac{dq_2(t)}{dt} + \theta q(t) = -Dt \quad t_1 \le t \le \mu, \quad q(\mu_-) = q(\mu_+) \qquad \ldots(42)$$

$$\frac{dq_3(t)}{dt} + \theta q(t) = -D\mu \quad \mu \le t \le T, \quad q(T) = 0 \qquad \ldots(43)$$

From Eq. (41), we have

$$q_1(t) = -D\delta\left[\frac{1}{2}t_1 t^2 - \frac{t^3}{3}\right] \qquad \ldots(44)$$

From Eq. (42), we have

$$q_2(t) = e^{-\alpha t^\beta} D\left[D_1 - \frac{1}{2}t^2 + \frac{\alpha}{\beta+2}t^{\beta+2}\right] \qquad \ldots(45)$$

Where,

$$D_1 = -\frac{\mu^2}{2} - \frac{\alpha\mu^{\beta+2}}{\beta+2} + \mu T - \frac{\alpha\mu}{\beta+1}\left(T^{\beta+1} - \mu^{\beta+1}\right)$$

From Eq. (43), we have

$$q_3(t) = e^{-\alpha t^\beta} D\mu\left\{D_2 - t - \frac{\alpha}{\beta+1}t^{\beta+1}\right\} \qquad \ldots(46)$$

Where,

$$D_2 = T + \frac{\alpha}{\beta+1}T^{\beta+1}$$

Total amount of deterioration during $[t_1, T]$

$$D_c = e^{-\alpha t_1^\beta}\left\{\int_{t_1}^{\mu} D t e^{\alpha t^\beta} dt + D\mu\int_{\mu}^{T} e^{\alpha t^\beta} dt - \int_{t_1}^{\mu} D t\,dt\right\} - (T - \mu)D\mu\int_{\mu}^{T} 1\,dt$$

$$D_c = D\left[\left(1 - \alpha t_1^\beta\right)\left(t_1^2 + \frac{\alpha t_1^{2+\beta}}{2+\beta} + D_3\right) - D_4\right] \qquad \ldots(47)$$

Where, $D_3 = \mu T - \mu^2 + \dfrac{\alpha\mu\left(T^{1+\beta} - \mu^{1+\beta}\right)}{1+\beta} - \dfrac{\mu^2}{2}$, $D_4 = \mu(T - \mu)^2$

Total holding cost during $[t_1, T]$ is $HC = \int_{t_1}^{T}(a + bt)q(t)\,dt$

$$HC = D\left[D_5 + D_6 + \frac{b}{8}t_1^4 + \frac{a}{6}t_1^3 - \frac{1}{2}bD_1t_1^2 - aD_1t_1 - D_7t_1^{\beta+4} - D_8t_1^{\beta+3} + D_9t_1^{\beta+2} + D_{10}t_1^{\beta+1} + D_{11}t_1^{2\beta+3} + D_{12}t_1^{2\beta+4}\right] \ldots(48)$$

Where,

$$D_5 = a\mu D_2(T-\mu) + \frac{\mu}{2}(bD_2 - a)(T^2-\mu^2) - \frac{b\mu}{3}(T^3-\mu^3) + \frac{\alpha^2 b\mu}{(\beta+1)(2\beta+3)}(T^{2\beta+3}-\mu^{2\beta+3})$$

$$+\frac{a\alpha^2\mu}{(\beta+1)(2\beta+2)}(T^{2\beta+2}-\mu^{2\beta+2}) + \frac{\alpha b\mu}{(\beta+3)}\left(1-\frac{1}{(\beta+1)}\right)(T^{\beta+3}-\mu^{\beta+3}) + \frac{\alpha\mu}{(\beta+2)}\left(a-\frac{a}{(\beta+1)}-bD_2\right)$$

$$(T^{\beta+2}-\mu^{\beta+2}) - \frac{a\alpha\mu D_2}{(\beta+1)}(T^{\beta+1}-\mu^{\beta+1}),\ D_6 = aD_1\mu + \frac{1}{2}bD_1\mu^2 - \frac{a}{6}\mu^3 - \frac{b}{8}\mu^4 - \frac{b\alpha^2}{(\beta+2)(2\beta+4)}\mu^{2\beta+4} + \frac{\alpha b}{(\beta+4)}$$

$$\left(\frac{1}{2}+\frac{1}{(\beta+2)}\right)\mu^{\beta+4} - \frac{a\alpha^2}{(\beta+2)(2\beta+3)}\mu^{2\beta+3} + \frac{a\alpha}{(\beta+3)}\left(\frac{1}{(\beta+2)}+\frac{1}{2}\right)\mu^{\beta+3} - \frac{\alpha bD_1}{(\beta+2)}\mu^{\beta+2} - \frac{a\alpha D_1}{(\beta+1)}\mu^{\beta+1},$$

$$D_7 = \frac{\alpha b}{(\beta+4)}\left(\frac{1}{(\beta+2)}+\frac{1}{2}\right),\ D_8 = \frac{a\alpha}{(\beta+3)}\left(\frac{1}{(\beta+2)}+\frac{1}{2}\right),\ D_9 = \frac{\alpha bD_1}{(\beta+2)},\ D_{10} = \frac{a\alpha D_1}{(\beta+1)},$$

$$D_{11} = \frac{a\alpha^2}{(\beta+2)(2\beta+3)},\ D_{12} = \frac{b\alpha^2}{(\beta+2)(2\beta+4)}$$

The shortage cost during $[0,t_1]$ is

$$S_c = \int_0^{t_1}(t_1-t)Dt\delta(t_1-t)\,dt = D\delta\int_0^{t_1} t(t_1-t)^2\,dt = \frac{D\delta}{12}t_1^4 \qquad \ldots(49)$$

The lost sales cost during $[0,t_1]$ is

$$LS = \int_0^{t_1}\left(1-\delta(t_1-t)\right)Dt\,dt = D\left(\frac{1}{2}t_1^2 - \frac{1}{6}\delta t_1^3\right) \qquad \ldots(50)$$

Ordering quantity during $[0,T]$ is

$$OQ_1' = \int_0^{t_1} Dt\delta(t_1-t)\,dt + e^{-\alpha t_1^\beta}\left\{\int_{t_1}^{\mu} Dte^{\alpha t^\beta}\,dt + D\mu\int_{\mu}^{T} e^{\alpha t^\beta}\,dt\right\}$$

$$OQ_1' = D\left[\frac{1}{6}\delta t_1^3 + (1-\alpha t_1^\beta)\left(\frac{1}{2}(\mu^2-t_1^2) + \frac{\alpha}{\beta+2}(\mu^{\beta+2}-t_1^{\beta+2}) + \mu\left(T-\mu+\frac{\alpha}{\beta+1}(T^{\beta+1}-\mu^{\beta+1})\right)\right)\right] \ldots(51)$$

Total cost during $[0,T]$ is the sum of deterioration cost, holding cost, shortage cost and lost sales cost is given by

$$TC_1'(t_1) = HC + S_c + LS + D_c$$

$$TC_1' = \mathrm{D}\left[\begin{array}{l}\left(\frac{b}{8}+\frac{\delta}{12}\right)t_1^4 + \frac{a-\delta}{6}t_1^3 + \frac{3-bD_1}{2}t_1^2 - aD_1t_1 + D_{12}t_1^{2\beta+4} + D_{11}t_1^{2\beta+3} - \frac{\alpha^2}{\beta+2}t_1^{2\beta+2} \\ -D_7t_1^{\beta+4} - D_8t_1^{\beta+3} + \left(D_9 - \alpha + \frac{\alpha}{\beta+2}\right)t_1^{\beta+2} + D_{10}t_1^{\beta+1} - D_3\alpha t_1^\beta + D_3 - D_4 + D_5 + D_6\end{array}\right] \ldots(52)$$

5.1.1. Solution:

$$\frac{\partial TC_1^{'}}{\partial t_1} = \mathrm{D}\begin{bmatrix} \left(\frac{b}{2}+\frac{\delta}{3}\right)t_1^{3} + \frac{a-\delta}{2}t_1^{2} + \left(3-bD_1\right)t_1 + D_{12}\left(2\beta+4\right)t_1^{2\beta+3} + D_{11}\left(2\beta+3\right)t_1^{2\beta+2} \\ -\frac{\alpha^{2}\left(2\beta+2\right)}{\beta+2}t_1^{2\beta+1} - D_7\left(\beta+4\right)t_1^{\beta+3} - D_8\left(\beta+3\right)t_1^{\beta+2} + \left(D_9 - \alpha + \frac{\alpha}{\beta+2}\right)\left(\beta+2\right)t_1^{\beta+1} \\ +D_{10}\left(\beta+1\right)t_1^{\beta} - D_3\alpha\beta t_1^{\beta-1} - aD_1 \end{bmatrix} \quad \ldots(53)$$

$$\frac{\partial^2 TC_1^{'}}{\partial t_1^{2}} = \mathrm{D}\begin{bmatrix} 3\left(\frac{b}{2}+\frac{\delta}{3}\right)t_1^{2} + \left(a-\delta\right)t_1 + D_{12}\left(2\beta+3\right)\left(2\beta+4\right)t_1^{2\beta+2} + D_{11}\left(2\beta+2\right)\left(2\beta+3\right)t_1^{2\beta+1} \\ -\frac{\alpha^{2}\left(2\beta+1\right)\left(2\beta+2\right)}{\beta+2}t_1^{2\beta} - D_7\left(\beta+3\right)\left(\beta+4\right)t_1^{\beta+2} - D_8\left(\beta+2\right)\left(\beta+3\right)t_1^{\beta+1} + 3 + \\ \left(D_9 - \alpha + \frac{\alpha}{\beta+2}\right)\left(\beta+1\right)\left(\beta+2\right)t_1^{\beta} + D_{10}\beta\left(\beta+1\right)t_1^{\beta-1} - D_3\alpha\beta\left(\beta-1\right)t_1^{\beta-2} - bD_1 \end{bmatrix} \quad \ldots(54)$$

Main objective is to minimize the total relevant cost for the inventory model starting without shortages. The essential condition to minimize the total relevant cost is $\frac{\partial TC_1^{'}}{\partial t_1} = 0$, we have

$$\left(\frac{b}{2}+\frac{\delta}{3}\right)t_1^{3} + \frac{a-\delta}{2}t_1^{2} + \left(3-bD_1\right)t_1 + D_{12}\left(2\beta+4\right)t_1^{2\beta+3} + D_{11}\left(2\beta+3\right)t_1^{2\beta+2} - \frac{\alpha^{2}\left(2\beta+2\right)}{\beta+2}t_1^{2\beta+1}$$
$$-D_7\left(\beta+4\right)t_1^{\beta+3} - D_8\left(\beta+3\right)t_1^{\beta+2} + \left(D_9 - \alpha + \frac{\alpha}{\beta+2}\right)\left(\beta+2\right)t_1^{\beta+1} + D_{10}\left(\beta+1\right)t_1^{\beta}$$
$$-D_3\alpha\beta t_1^{\beta-1} - aD_1 = 0 \quad \ldots(55)$$

Using the software Mathematica, we can calculate the optimal value of t_1 by Eq. (55) and the optimal value $TC_1^{'}\left(t_1\right)$ of the total relevant cost is determined by Eq. (52). The optimal value of t_1 satisfy the sufficient condition for minimizing total relevant cost $TC_1^{'}\left(t_1\right)$ is

$$\frac{\partial^2 TC_1^{'}}{\partial t_1^{2}} > 0 \quad \ldots(56)$$

The sufficient condition is satisfied.

5.1.2. Numerical Example:

Let us consider

$a = \$1/unit, b = \$0.5/unit, S_c = \$0.5/unit, D_c = \$5/unit, \alpha = 0.1, \beta = 2, T = 1year, D = 100,$
$\delta = 1, \mu = 0.5\ year$
$D_1 = 0.36, D_2 = 1.03, D_3 = 0.14, D_4 = -0.25, D_5 = 0.096, D_6 = 0.27, D_7 = 0.00625, D_8 = 0.015,$
$D_9 = 0.0045, D_{10} = 0.006, D_{11} = 0.00035, D_{12} = 0.00015$

Thus, the optimal value of t_1 is $t_1^{*} \rightarrow 0.035 < \mu$. The optimum ordering quantity is $OQ^{'*} = 39.05$. The minimum relevant cost is $TC_1^{'*} = 33.9$.

5.1.3.When Holding Cost is a Quadratic Function:

We have

$$HC = \int_{t_1}^{T} \left(a + bt + ct^2 \right) q\left(t \right) dt$$

$$HC = D\left[\begin{array}{l} \frac{c}{10}t_1^{5} + \frac{b}{8}t_1^{4} + D_7 t_1^{3} - \frac{1}{2}bD_1 t_1^{2} - aD_1 t_1 + D_5 + D_6 - D_8 t_1^{\beta+3} + D_9 t_1^{\beta+1} \\ + D_{10} t_1^{2\beta+3} - D_{11} t_1^{\beta+4} + \frac{\alpha b D_1}{(\beta+2)} t_1^{\beta+2} + D_{12} t_1^{2\beta+4} - D_{13} t_1^{\beta+5} + D_{14} t_1^{2\beta+5} \end{array} \right] \quad \ldots(57)$$

Where,

$$D_5 = \mu\left[\begin{array}{l} aD_2 (T - \mu) + \frac{(bD_2 - a)}{2}\left(T^2 - \mu^2\right) + \frac{cD_2 - b}{3}\left(T^3 - \mu^3\right) - \frac{c}{4}\left(T^4 - \mu^4\right) + \frac{c\alpha^2}{(\beta+1)(2\beta+4)}\left(T^{2\beta+4} - \mu^{2\beta+4}\right) \\ + \frac{\alpha^2 b}{(\beta+1)(2\beta+3)}\left(T^{2\beta+3} - \mu^{2\beta+3}\right) + \frac{a\alpha^2}{(\beta+1)(2\beta+2)}\left(T^{2\beta+2} - \mu^{2\beta+2}\right) + \frac{c\alpha}{(\beta+4)}\left(1 - \frac{1}{(\beta+1)}\right)\left(T^{\beta+4} - \mu^{\beta+4}\right) \\ + \frac{\alpha}{(\beta+3)}\left(b - cD_2 - \frac{b}{(\beta+1)}\right)\left(T^{\beta+3} - \mu^{\beta+3}\right) + \frac{\alpha}{(\beta+2)}\left(a - bD_2 - \frac{a}{(\beta+1)}\right)\left(T^{\beta+2} - \mu^{\beta+2}\right) - \frac{a\alpha D_2}{(\beta+1)}\left(T^{\beta+1} - \mu^{\beta+1}\right) \end{array} \right],$$

$$D_7 = \frac{a}{6} - \frac{cD_1}{3},\ D_8 = \frac{\alpha}{(\beta+3)}\left(\frac{a}{2} + \frac{a}{(\beta+2)} + cD_1\right), D_9 = \frac{a\alpha D_1}{(\beta+1)}, D_{10} = \frac{a\alpha^2}{(\beta+2)(2\beta+3)},$$

$$D_{11} = \frac{\alpha b}{(\beta+4)}\left(\frac{1}{2} + \frac{1}{(\beta+2)}\right), D_{12} = \frac{b\alpha^2}{(\beta+2)(2\beta+4)}, D_{13} = \frac{c\alpha}{(\beta+5)}\left(\frac{1}{2} + \frac{1}{(\beta+2)}\right),\ D_{14} = \frac{c\alpha^2}{(\beta+2)(2\beta+5)}$$

$$D_6 = aD_1\mu + \frac{1}{2}bD_1\mu^2 + \left(\frac{cD_1}{3} - \frac{a}{6}\right)\mu^3 - \frac{b}{8}\mu^4 - \frac{c}{10}\mu^5 - \frac{c\alpha^2\mu^{2\beta+5}}{(\beta+2)(2\beta+5)} - \frac{b\alpha^2}{(\beta+2)(2\beta+4)}\mu^{2\beta+4}$$
$$- \frac{a\alpha^2}{(\beta+2)(2\beta+3)}\mu^{2\beta+3} + \frac{c\alpha}{(\beta+5)}\left(\frac{1}{2} + \frac{1}{(\beta+2)}\right)\mu^{\beta+5} + \frac{\alpha b}{(\beta+4)}\left(\frac{1}{2} + \frac{1}{(\beta+2)}\right)\mu^{\beta+4} + \frac{\alpha}{(\beta+3)}$$
$$\left(\frac{a}{2} + \frac{a}{(\beta+2)} - cD_1\right)\mu^{\beta+3} - \frac{\alpha b D_1}{(\beta+2)}\mu^{\beta+2} - \frac{a\alpha D_1}{(\beta+1)}\mu^{\beta+1},$$

Total cost is the sum of quadratic holding cost, shortage cost, deterioration cost and lost sales cost during $\left[0, T\right]$

$$TC_1^{'}\left(t_1\right) = HC + S_c + LS + D_c$$

$$TC_1^{'} = \mathrm{D}\left[\begin{array}{l} \frac{c}{10}t_1^{5} + \left(\frac{b}{8} + \frac{\delta}{12}\right)t_1^{4} + \left(D_7 - \frac{\delta}{6}\right)t_1^{3} + \frac{1}{2}\left(3 - bD_1\right)t_1^{2} - aD_1 t_1 + D_{14} t_1^{2\beta+5} + D_{12} t_1^{2\beta+4} + D_{10} t_1^{2\beta+3} - \frac{\alpha^2}{2+\beta} t_1^{2\beta+2} \\ - D_{13} t_1^{\beta+5} - D_{11} t_1^{\beta+4} - D_8 t_1^{\beta+3} + \frac{\alpha}{\beta+2}\left(bD_1 - \beta - 1\right)t_1^{\beta+2} + D_9 t_1^{\beta+1} - D_3 \alpha t_1^{\beta} + D_3 - D_4 + D_5 + D_6 \end{array} \right] \quad \ldots(58)$$

5.1.4. Solution:

$$\frac{\partial TC_1^{'}}{\partial t_1} = D\begin{bmatrix} \frac{c}{2}t_1^{4} + \left(\frac{b}{2}+\frac{\delta}{3}\right)t_1^{3} + \left(3D_7 - \frac{\delta}{2}\right)t_1^{2} + (3 - bD_1)t_1 - aD_1 + D_{14}(2\beta+5)t_1^{2\beta+4} + D_{12}(2\beta+4)t_1^{2\beta+3} + \\ D_{10}(2\beta+3)t_1^{2\beta+2} - \frac{\alpha^2}{2+\beta}(2\beta+2)t_1^{2\beta+1} - D_{13}(\beta+5)t_1^{\beta+4} - D_{11}(\beta+4)t_1^{\beta+3} - D_8(\beta+3)t_1^{\beta+2} + \\ \alpha(bD_1 - \beta - 1)t_1^{\beta+1} + D_9(\beta+1)t_1^{\beta} - D_3\alpha\beta t_1^{\beta-1} \end{bmatrix} \quad \ldots(59)$$

$$\frac{\partial^2 TC_1^{'}}{\partial t_1^2} = D\begin{bmatrix} 2ct_1^{3} + 3\left(\frac{b}{2}+\frac{\delta}{3}\right)t_1^{2} + 2\left(3D_7 - \frac{\delta}{2}\right)t_1 + (3 - bD_1) + D_{14}(2\beta+4)(2\beta+5)t_1^{2\beta+3} + D_{12}(2\beta+3)(2\beta+4)t_1^{2\beta+2} \\ + D_{10}(2\beta+2)(2\beta+3)t_1^{2\beta+1} - \frac{2\alpha^2}{\beta+2}(\beta+1)(2\beta+1)t_1^{2\beta} - D_{13}(\beta+4)(\beta+5)t_1^{\beta+3} - D_{11}(\beta+3)(\beta+4)t_1^{\beta+2} \\ - D_8(\beta+2)(\beta+3)t_1^{\beta+1} + \alpha(\beta+1)(bD_1 - \beta - 1)t_1^{\beta} + D_9\beta(\beta+1)t_1^{\beta-1} - D_3\alpha\beta(\beta-1)t_1^{\beta-2} \end{bmatrix} \ldots(60)$$

Main objective is to minimize the total relevant cost for the inventory model starting without shortages. The essential condition to minimize the total relevant cost is $\frac{\partial TC_1^{'}}{\partial t_1} = 0$, we have

$$\frac{c}{2}t_1^{4} + \left(\frac{b}{2}+\frac{\delta}{3}\right)t_1^{3} + \left(3D_7 - \frac{\delta}{2}\right)t_1^{2} + (3 - bD_1)t_1 - aD_1 + D_{14}(2\beta+5)t_1^{2\beta+4} + D_{12}(2\beta+4)$$
$$t_1^{2\beta+3} + D_{10}(2\beta+3)t_1^{2\beta+2} - \frac{\alpha^2}{2+\beta}(2\beta+2)t_1^{2\beta+1} - D_{13}(\beta+5)t_1^{\beta+4} - D_{11}(\beta+4)t_1^{\beta+3} - D_8$$
$$(\beta+3)t_1^{\beta+2} + \alpha(bD_1 - \beta - 1)t_1^{\beta+1} + D_9(\beta+1)t_1^{\beta} - D_3\alpha\beta t_1^{\beta-1} = 0 \quad \ldots(61)$$

Using the software Mathematica, we can calculate the optimal value of t_1 by Eq. (61) and the optimal value $TC_1^{'}(t_1)$ of the total relevant cost is determined by Eq. (58). The optimal value of t_1 satisfy the sufficient condition for minimizing total relevant cost $TC_1^{'}(t_1)$ is

$$\frac{\partial^2 TC_1^{'}}{\partial t_1^2} > 0 \quad \ldots(62)$$

The sufficient condition is satisfied.

5.1.5. Numerical Example:

Let us consider

$a = \$1/unit, b = \$0.5/unit, c = \$1/unit, S_c = \$0.5/unit, D_c = \$5/unit, \alpha = 0.1, \beta = 2, T = 1 year, D = 100,$
$\delta = 1, \mu = 0.5\ year$
$D_1 = 0.36, D_2 = 1.03, D_3 = 0.14, D_4 = -0.25, D_5 = 0.112, D_6 = 0.3, D_7 = 0.046, D_8 = 0.022, D_9 = 0.012,$
$D_{10} = 0.0003, D_{11} = 0.00625, D_{12} = 0.00015, D_{13} = 0.01, D_{14} = 0.0003$

Thus, the optimal value of t_1 is $t_1^{*} \to 0.035 < \mu$. The optimum ordering quantity is $OQ^{'*} = 39.05$. The minimum relevant cost is $TC_1^{'*} = 33.9$.

5.2. Case (ii) $t_1 \geq \mu$

Therefore, the inventory $I(t)$ is described by the system of differential equations during $[0, T]$

$$\frac{dq_1(t)}{dt} = -Dt\delta(t_1 - t), \quad 0 \le t \le \mu, \quad q(0) = 0 \qquad \ldots(63)$$

$$\frac{dq_2(t)}{dt} = -D\mu\delta(t_1 - t), \quad \mu \le t \le t_1, \quad q(\mu_-) = q(\mu_+) \qquad \ldots(64)$$

$$\frac{dq_3(t)}{dt} + \theta q(t) = -D\mu, \quad t_1 \le t \le T, \quad q(T) = 0 \qquad \ldots(65)$$

From Eq.(63), we have

$$q_1(t) = -D\delta\left[\frac{1}{2}t_1 t^2 - \frac{t^3}{3}\right], \quad 0 \le t \le \mu \qquad \ldots(66)$$

From Eq.(64), we have

$$q_2(t) = -D\delta\mu\left[-\frac{1}{2}t_1\mu + \frac{\mu^2}{6} + t_1 t - \frac{1}{2}t^2\right], \mu \le t \le t_1 \qquad \ldots(67)$$

FromEq.(65), we have

$$q_3(t) = e^{-\alpha t^\beta} D\mu\left(D_2 - t - \frac{\alpha}{\beta+1}t^{\beta+1}\right), \quad t_1 \le t \le T \qquad \ldots(68)$$

Where,

$$D_2 = T + \frac{\alpha}{\beta+1}T^{\beta+1}$$

Total amount of deterioration during $[t_1, T]$

$$D_c = e^{-\alpha t_1^\beta} D\mu\int_{t_1}^{T} e^{\alpha t^\beta} dt - D\mu\int_{t_1}^{T} dt = D\mu\left(\left(1-\alpha t_1^\beta\right)\left(D_2 - t_1 - \frac{\alpha}{\beta+1}t_1^{\beta+1}\right) - (T - t_1)\right) \ldots(69)$$

Total holding cost during $[t_1, T]$ is

$$HC = \int_{t_1}^{T}(a+bt)q_3(t)\,dt$$

$$= D\mu\left[F_1 - aD_2t_1 + F_2t_1^2 + \frac{b}{3}t_1^3 + F_3t_1^{\beta+3} + F_4t_1^{\beta+2} + F_5t_1^{\beta+1} + F_6t_1^{2\beta+2} - F_7t_1^{2\beta+3}\right] \qquad \ldots(70)$$

Where,

$$F_1 = \frac{b\alpha^2}{(\beta+1)(2\beta+3)}T^{2\beta+3} + \alpha\left(\frac{a\alpha}{2(\beta+1)^2} - \frac{bD_2}{(\beta+2)}\right)T^{2\beta+2} + \frac{b\alpha}{(\beta+3)}\left(1 - \frac{1}{(\beta+1)}\right)T^{\beta+3} + \frac{a\alpha}{(\beta+2)}$$
$$\left(1 - \frac{1}{(\beta+1)}\right)T^{\beta+2} - \frac{a\alpha D_2}{(\beta+1)}T^{\beta+1} - \frac{b}{3}T^3 + \frac{1}{2}(bD_2 - a)T^2 + aD_2T, F_2 = \frac{1}{2}(a - bD_2), F_3 = \frac{b\alpha}{(\beta+3)}$$
$$\left(\frac{1}{(\beta+1)} - 1\right), F_4 = \frac{a\alpha}{(\beta+2)}\left(\frac{1}{(\beta+1)} - 1\right), F_5 = \frac{a\alpha D_2}{(\beta+1)}, F_6 = \alpha\left(\frac{bD_2}{(\beta+2)} - \frac{a\alpha}{2(\beta+1)^2}\right), F_7 = \frac{b\alpha^2}{(\beta+1)(2\beta+3)}$$

The shortage cost during $[0, t_1]$ is

$$S_c = \int_0^{\mu} q_1(t)\,dt + \int_{\mu}^{t_1} q_2(t)\,dt = D\delta\left[\frac{\mu}{3}t_1^3 - \frac{\mu^2}{2}t_1^2 + \frac{\mu^3}{3}t_1 - \frac{\mu^4}{12}\right] \qquad \ldots(71)$$

The lost sales cost during $[0, t_1]$ is

$$LS = \int_0^{\mu} Dt\left(1-\delta(t_1-t)\right)dt + D\mu\int_{\mu}^{t_1}\left(1-\delta(t_1-t)\right)dt = -\frac{1}{6}\mu D\left(F_9 - F_{10}t_1 + 3\delta t_1^2\right) \quad \ldots(72)$$

Where,

$$F_9 = \mu(3+\delta\mu), F_{10} = 3(2+\delta\mu)$$

Ordering quantity during $[0, T]$ is

$$OQ = \int_0^{\mu} Dt\left(1-\delta(t_1-t)\right)dt + D\mu\int_{\mu}^{t_1}\delta(t_1-t)\,dt + e^{-\alpha t_1^{\beta}}D\mu\int_{t_1}^{T}e^{\alpha t^{\beta}}dt$$

$$OQ = D\mu\left[\frac{\alpha^2}{\beta+1}t_1^{2\beta+1} + \frac{\alpha\beta}{\beta+1}t_1^{\beta+1} - D_2\alpha t_1^{\beta} + \frac{1}{2}\delta t_1^2 - \left(\frac{3}{2}\mu\delta+1\right)t_1 + \frac{5}{6}\delta\mu^2 + \frac{1}{2}\mu + D_2\right] \qquad \ldots(73)$$

Total cost is the sum of deterioration cost, holding cost, shortage cost and lost sales cost during $[0, T]$

$$TC_1'(t_1) = HC + S_c + LS + D_c$$

$$TC_2' = D\mu\left[\begin{array}{l}\frac{1}{3}(b+\delta)t_1^3 + \frac{1}{2}(2F_2-\delta-\delta\mu)t_1^2 + \frac{1}{6}\left(F_{10}+2\delta\mu^2-6aD_2\right)t_1 - D_2\alpha t_1^{\beta} + \left(F_5+\frac{\alpha\beta}{\beta+1}\right)\\ t_1^{\beta+1} + F_4t_1^{\beta+2} + F_3t_1^{\beta+3} + \frac{\alpha^2}{\beta+1}t_1^{2\beta+1} + F_6t_1^{2\beta+2} - F_7t_1^{2\beta+3} - \frac{\delta\mu^3}{12} - \frac{1}{6}F_9 + D_2 + F_1 - T\end{array}\right] \ldots(74)$$

5.2.1. Solution:

$$\frac{\partial TC_2'}{\partial t_1} = D\mu\left[\begin{array}{l}(b+\delta)t_1^2 + (2F_2-\delta-\delta\mu)t_1 + \frac{1}{6}\left(F_{10}+2\delta\mu^2-6aD_2\right) - D_2\alpha\beta t_1^{\beta-1} + \\ \left(F_5+\frac{\alpha\beta}{\beta+1}\right)(\beta+1)t_1^{\beta} + F_4(\beta+2)t_1^{\beta+1} + F_3(\beta+3)t_1^{\beta+2} + \frac{\alpha^2}{\beta+1}(2\beta+1)t_1^{2\beta} \\ +F(2\beta+2)_6 t_1^{2\beta+1} - F_7(2\beta+3)t_1^{2\beta+2}\end{array}\right] \ldots(75)$$

$$\frac{\partial^2 TC_2'}{\partial t_1^2} = D\mu\left[\begin{array}{l}2(b+\delta)t_1 + 2F_2 - \delta - \delta\mu - D_2\alpha\beta(\beta-1)t_1^{\beta-2} + \beta\left(F_5(\beta+1)+\alpha\beta\right)t_1^{\beta-1} \\ +F_4(\beta+1)(\beta+2)t_1^{\beta} + F_3(\beta+2)(\beta+3)t_1^{\beta+1} + \frac{2\alpha^2\beta(2\beta+1)}{\beta+1}t_1^{2\beta-1} + \\ F_6(2\beta+1)(2\beta+2)t_1^{2\beta} - 2F_7(\beta+1)(2\beta+3)t_1^{2\beta+1}\end{array}\right] \ldots(76)$$

Main objective is to minimize the total relevant cost for the inventory model starting without shortages. The essential condition to minimize the total relevant cost is $\frac{\partial TC_2'}{\partial t_1} = 0$, we have

$$(b+\delta)t_1^{2} + (2F_2 - \delta - \delta\mu)t_1 + \frac{1}{6}\left(F_{10} + 2\delta\mu^{2} - 6aD_2\right) - D_2\alpha\beta t_1^{\beta-1} + \left(F_5 + \frac{\alpha\beta}{\beta+1}\right)(\beta+1)t_1^{\beta} +$$
$$F_4(\beta+2)t_1^{\beta+1} + F_3(\beta+3)t_1^{\beta+2} + \frac{\alpha^{2}}{\beta+1}(2\beta+1)t_1^{2\beta} + F(2\beta+2)_6\, t_1^{2\beta+1} - F_7(2\beta+3)t_1^{2\beta+2} = 0 \quad \ldots(77)$$

Using the software Mathematica, we can calculate the optimal value of t_1 by Eq. (77) and the optimal value $TC_1^{'}(t_1)$ of the total relevant cost is determined by Eq. (74). The optimal value of t_1 satisfy the sufficient condition for minimizing total relevant cost $TC_1^{'}(t_1)$ is

$$\frac{\partial^{2} TC_2^{'}}{\partial t_1^{2}} > 0 \qquad \ldots(78)$$

The sufficient condition is satisfied.

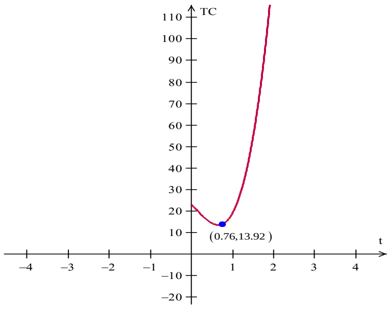

Fig.4. Graphical representation of $TC_2^{'}$ and t_1.

5.2.2. Numerical Example:

Let us consider

$a = \$1/unit, b = \$0.5/unit, S_c = \$0.5/unit, D_c = \$5/unit, \alpha = 0.1, \beta = 2, T = 1 year, D = 100, \delta = 1,$
$\mu = 0.5\ year$
$D_2 = 1.03, F_1 = 0.597, F_2 = 0.24, F_3 = -0.0067, F_4 = -0.017, F_5 = 0.034, F_6 = 0.012, F_7 = 0.00023,$
$F_9 = 1.75, F_{10} = 7.5$

Thus, the optimal value of t_1 is $t_1^{*} \to 0.76 > \mu$. The optimum ordering quantity is $OQ^{'*} = 20.9$. The minimum relevant cost is $TC_2^{'*} = 13.92$.

5.2.3. SensitivityAnalysis:

To know, how the optimal solution is affected by the values ofconstraints, we derive the sensitivity analysis for some constraints. The specific values of some constraintsare increased or decreased by $+25\%$, -25% and $+50\%$, -50% .Thus,we computethe values of t_1^{*} and TC_1^{*} with the help of increased or decreased values of S_c and D_c .The result of minimum relevant cost is existing in the following table 3.

Table: 3

Parameters	Actual Values	+50% Increased	-50% Decreased	+25% Increased	-25% Decreased
S_c	0.5	0.75	0.25	0.625	0.375
D_c	5	7.5	2.5	6.25	3.75
t_1^{*}	0.76	0.69	0.84	0.72	0.79
TC_1^{*}	13.92	19.71	7.5	16.88	10.82
OQ^{*}	20.9	24.6	16.97	22.98	19.37

From the result of above table, we observe that total relevant cost and ordering quantity is much affected by deterioration cost and shortage cost, and other parameters are less sensitive.

5.2.4. When Holding Cost is a Quadratic Function:
We have

$$HC = \int_{t_1}^{T} \left(a + bt + ct^2\right) q_3(t)\, dt$$

$$= D\mu \begin{bmatrix} F_1 - aD_2 t_1 + F_2 t_1^2 + F_3 t_1^3 + \dfrac{c}{4} t_1^4 + F_4 t_1^{\beta+4} + F_5 t_1^{\beta+3} + F_6 t_1^{\beta+2} + \dfrac{a\alpha D_2}{(\beta+1)} t_1^{\beta+1} \\ + F_7 t_1^{2\beta+2} - F_8 t_1^{2\beta+4} - F_9 t_1^{2\beta+3} \end{bmatrix} \qquad \ldots(79)$$

Where,

$$F_1 = \frac{c\alpha^2}{(\beta+1)(2\beta+4)} T^{2\beta+4} + \frac{b\alpha^2}{(\beta+1)(2\beta+3)} T^{2\beta+3} + \alpha\left(\frac{a\alpha}{2(\beta+1)^2} - \frac{bD_2}{(\beta+2)}\right) T^{2\beta+2} + \frac{c\alpha}{(\beta+4)}\left(1 - \frac{1}{(\beta+1)}\right) T^{\beta+4} + aD_2 T$$

$$+ \frac{b\alpha}{(\beta+3)}\left(1 - D_2 - \frac{1}{(\beta+1)}\right) T^{\beta+3} + \frac{a\alpha}{(\beta+2)}\left(1 - \frac{1}{(\beta+1)}\right) T^{\beta+2} - \frac{a\alpha D_2}{(\beta+1)} T^{\beta+1} - \frac{c}{4} T^4 + \frac{1}{3}(cD_2 - b) T^3 + \frac{1}{2}(bD_2 - a) T^2$$

$$F_2 = \frac{1}{2}(a - bD_2),\ F_3 = \frac{1}{3}(b - cD_2),\ F_4 = \frac{c\alpha}{(\beta+4)}\left(\frac{1}{(\beta+1)} - 1\right),\ F_5 = \frac{\alpha}{(\beta+3)}\left(cD_2 - b + \frac{b}{(\beta+1)}\right),$$

$$F_6 = \frac{a\alpha}{(\beta+2)}\left(\frac{1}{(\beta+1)} - 1\right),\ F_7 = \alpha\left(\frac{bD_2}{(\beta+2)} - \frac{a\alpha}{2(\beta+1)^2}\right),\ F_8 = \frac{c\alpha^2}{(\beta+1)(2\beta+4)},\ F_9 = \frac{b\alpha^2}{(\beta+1)(2\beta+3)}$$

Total cost is the sum of quadratic holding cost, shortage cost, deterioration cost and lost sales cost during $[0, T]$

$$TC_1'(t_1) = HC + S_c + LS + D_c$$

$$TC_2' = D\mu \begin{bmatrix} \dfrac{c}{4} t_1^4 + \left(F_3 + \dfrac{\delta}{3}\right) t_1^3 + \dfrac{1}{2}(2F_2 - \delta - \delta\mu) t_1^2 + \dfrac{1}{6}\left(F_{10} + 2\delta\mu^2 - 6aD_2\right) t_1 - D_2 \alpha t_1^{\beta} + \dfrac{\alpha}{\beta+1}(\beta + aD_2) t_1^{\beta+1} \\ + F_6 t_1^{\beta+2} + F_5 t_1^{\beta+3} + F_4 t_1^{\beta+4} + \dfrac{\alpha^2}{\beta+1} t_1^{2\beta+1} + F_7 t_1^{2\beta+2} - F_9 t_1^{2\beta+3} - F_8 t_1^{2\beta+4} - \dfrac{\delta\mu^3}{12} - \dfrac{F_9}{6} + D_2 + F_1 - T \end{bmatrix} \ldots(80)$$

5.2.5. Solution:

$$\frac{\partial TC_2'}{\partial t_1} = D\mu \begin{bmatrix} ct_1^3 + (3F_3+\delta)t_1^2 + (2F_2-\delta-\delta\mu)t_1 + \frac{1}{6}\left(F_{10}+2\delta\mu^2-6aD_2\right) - D_2\alpha\beta t_1^{\beta-1} \\ +\alpha(\beta+aD_2)t_1^{\beta} + F_6(\beta+2)t_1^{\beta+1} + F_5(\beta+3)t_1^{\beta+2} + F_4(\beta+4)t_1^{\beta+3} \\ +\frac{\alpha^2(2\beta+1)}{\beta+1}t_1^{2\beta} + F_7(2\beta+2)t_1^{2\beta+1} - F_9(2\beta+3)t_1^{2\beta+2} - F_8(2\beta+4)t_1^{2\beta+3} \end{bmatrix} \quad \ldots(81)$$

$$\frac{\partial^2 TC_2'}{\partial t_1^2} = D\mu \begin{bmatrix} 3ct_1^2 + 2(3F_3+\delta)t_1 + (2F_2-\delta-\delta\mu) - D_2\alpha\beta(\beta-1)t_1^{\beta-2} + \alpha\beta(\beta+aD_2)t_1^{\beta-1} + \\ F_6(\beta+1)(\beta+2)t_1^{\beta} + F_5(\beta+2)(\beta+3)t_1^{\beta+1} + F_4(\beta+3)(\beta+4)t_1^{\beta+2} + \frac{\alpha^2 2\beta(2\beta+1)}{\beta+1} \\ t_1^{2\beta-1} + F_7(2\beta+1)(2\beta+2)t_1^{2\beta} - F_9(2\beta+2)(2\beta+3)t_1^{2\beta+1} - F_8(2\beta+3)(2\beta+4)t_1^{2\beta+2} \end{bmatrix} \ldots(82)$$

Main objective is to minimize the total relevant cost for the inventory model starting without shortages. The essential condition to minimize the total relevant cost is $\frac{\partial TC_2'}{\partial t_1} = 0$, we have

$$ct_1^3 + (3F_3+\delta)t_1^2 + (2F_2-\delta-\delta\mu)t_1 + \frac{1}{6}\left(F_{10}+2\delta\mu^2-6aD_2\right) - D_2\alpha\beta t_1^{\beta-1} + \alpha(\beta+aD_2)t_1^{\beta}$$

$$+F_6(\beta+2)t_1^{\beta+1} + F_5(\beta+3)t_1^{\beta+2} + F_4(\beta+4)t_1^{\beta+3} + \frac{\alpha^2(2\beta+1)}{\beta+1}t_1^{2\beta} + F_7(2\beta+2)t_1^{2\beta+1}$$

$$-F_9(2\beta+3)t_1^{2\beta+2} - F_8(2\beta+4)t_1^{2\beta+3} = 0 \quad \ldots(83)$$

Using the software Mathematica, we can calculate the optimal value of t_1 by Eq. (83) and the optimal value $TC_1'(t_1)$ of the total relevant cost is determined by Eq. (80). The optimal value of t_1 satisfy the sufficient condition for minimizing total relevant cost $TC_1'(t_1)$ is

$$\frac{\partial^2 TC_2'}{\partial t_1^2} > 0 \quad \ldots(84)$$

The sufficient condition is satisfied.

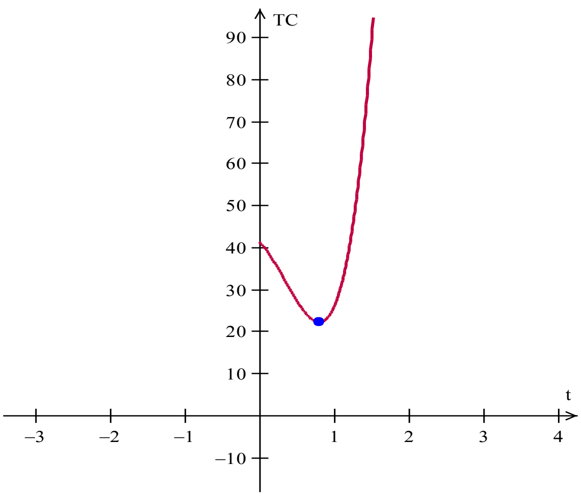

Fig.5. Graphical representation of TC_2' and t_1.

5.2.6. Numerical Example:

Let us consider

$a = \$1/unit, b = \$0.5/unit, S_c = \$0.5/unit, D_c = \$5/unit, \alpha = 0.1, \beta = 2, T = 1 year, D = 100,$

$\delta = 1, \mu = 0.5\ year$

$D_2 = 1.03, F_1 = 0.682, F_2 = 0.24, F_3 = -0.176, F_4 = -0.011, F_5 = 0.014, F_6 = -0.016, F_7 = 0.012,$

$F_8 = 0.0004, F_9 = 0.00024, F_{10} = 7.5$

Thus, the optimal value of t_1 is $t_1^* \rightarrow 0.8 > \mu$. The optimum ordering quantity is $OQ^{'*} = 20.9$. The minimum relevant cost is $TC_2^{'*} = 22.15$.

5.2.7. SensitivityAnalysis:

To know, how the optimal solution is affected by the values of constraints, we derive the sensitivity analysis for some constraints. The specific values of some constraints areincreased or decreased by $+25\%$, -25% and $+50\%$, -50% .Thus,we compute the values of t_1^* and TC_1^* with the help of increased or decreased values of S_c and D_c. The result of the minimum relevant cost is existing in the following table 4.

Table: 4

Parameters	Actual Values	+50% Increased	-50% Decreased	+25% Increased	-25% Decreased
S_c	0.5	0.75	0.25	0.625	0.375
D_c	5	7.5	2.5	6.25	3.75
t_1^*	0.8	0.74	0.88	0.77	0.84
TC_1^*	22.15	31.99	12.13	27.48	17.004
OQ^*	20.9	24.6	16.97	22.98	19.37

From the result of above table, we observe that total relevant cost and ordering quantity is much affected by deterioration cost and shortage cost, and other parameters are less sensitive.

VI. Conclusion

We presented an order level inventory model for deteriorating items with two parameters Weibull deterioration. The model developed under two replenishment policies: (i) With no shortages and (ii) With shortages which are partially backlogged. We considered ramp-type demand rate and time varying linear and quadratic holding costs, and found that total relevant cost with linear holding cost is less than total relevant cost with quadratic holding cost. Therefore, the linear time-dependent holding cost is more realistic than quadratic time-dependent holding cost. The proposed model can be extended in numerous ways like permissible delay in payments, time value of money,quantity discounts etc.

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References

[1] S. P. Aggarwal.and C. K.Jaggi. Ordering policies of deteriorating items under permissible delay inpayments. Journal of Operational Research Society, 36: 658-662, 1995.

[2] T. Chakrabarti. and K. S. Chaudhuri. An EOQ model for deteriorating items with a linear trend in demandand shortages in all cycles. International Journal of Production Economics, 49: 205-213, 1997.

[3] H. J. Chang. and C. Y. Dye. An EOQ model for deteriorating items with time varying demand and partialbacklogging. Journal of Operational Research Society, 50: 1176-1182, 1999.

[4] R. P. Covert. andG. C. Phillip. An EOQ model for deteriorating items with weibull distributiondeterioration.AIIE Transaction, 5: 323-326, 1973.

[5] W. C. Donaldson. Inventory replenishment policy for a linear trend in demand-an analytical solution.Operational Research Quarterly, 28: 663-670, 1977.

[6] H. Emmons. A replenishment model for radioactive nuclide generators.Management Science, 14: 263-273, 1968.

[7] P. M. Ghare. andG. F. Schrader. A model for exponentially decaying inventories.Journal of IndustrialEngineering, 14: 238-243, 1963.

[8] B. C. Giri.A. K. Jalan.and K. S. Chaudhuri.Economic order quantity model with Weibull deteriorationdistribution, shortage and ramp-type demand. International Journal of Systems Science, 34(4): 237-243, 2003.

[9] S. K. Ghosh. andK. S. Chaudhuri. An order-level inventory model for a deteriorating item with weibulldistribution deterioration, time-quadratic demand and shortages, Advanced Modeling and Optimization, 6: 21-35, 2004.

[10] S. K. Goyal. Economic order quantity model under conditions of permissible delay in payment.Journal ofOperational Research Society, 36: 335-338, 1985.

[11] Goyal et al. The finite horizon trended inventory replenishment problem with shortages.Journal ofOperational Research Society, 43: 1173-1178, 1992.

[12] S. K. Goyal. andB. C. Giri. Recent trends in modeling of deteriorating inventory.European Journalof Operational Research, 134: 1-16, 2001.

[13] C. W. Haley. andR. C. Higgin. Inventory policy and trade credit financing.Management Science, 20: 464-471, 1973.

[14] M. Hariga. The inventory replenishment problem with continuous linear trend in demand.Computers andIndustrial Engineering, 24: 143-150, 1993.

[15] M. Hariga. Optimal EOQ models for deteriorating items with time-varying demand. Journal of OperationalResearch Society, 47: 1228-1246, 1996.

[16] M. Hariga. andA. Alyan.A lot sizing heuristic for deteriorating items with shortages in growing anddeclining markets.Computers and Operations Research, 24: 1075-1083, 1997.

[17] M. Hariga. andL. Benkherouf. Optimal and heuristic inventory replenishment models for deterioratingitems with exponential time-varying demand.European Journal of Operational Research, 79: 123-137, 1994.

[18] T. M. Murdeshwar. Inventory replenishment policy for linearly increasing demand considering shortages-an optimal solution.Journal of Operational Research Society, 39: 687-692, 1988.

[19] M. Roy., S. Sana.andK. S. Chaudhuri. An EOQ model for imperfect quality products with partialbacklogging-a comparative study.International Journal of Services and Operations management, 83-110, 2011a.

[20] M. Roy., S. Sana.andK. S. Chaudhuri. An economic order quantity model for imperfect quality items withpartial backlogging.International Journal of Systems Science, 42: 1409-1419, 2011b.

[21] M. Roy., S. Sana.andK. S. Chaudhuri. An optimal shipment strategy for imperfect items in stock-outsituation.Mathematical and Computer Modeling, in press, 2011c.

[22] S. Sana. Optimal selling price and lot-size with time varying deterioration and partial backlogging.AppliedMathematics and Computation, 217: 185-194, 2010a.

[23] B. R. Sarkar. et al. An ordering for deteriorating items with allowable shortage and permissible delay inpayment.Journal of Operational Research Society, 48: 826-833, 1997.

[24] Y. K. Shah. andM. C. Jaiswal. An order-level inventory model for asystem with constant rate of deterioration.Opsearch, 14: 174-184, 1977.

[25] E. A. Silver. andH. C. Meal. A simple modification of the EOQ for the case of a varying demand rate.Production of Inventory Management, 10: 52-65, 1969.

[26] K. Skouri, I. Konstantaras, S. Papachristos, andI. Ganas.Inventory models with ramp type demand rate,partial backlogging and Weibull deterioration rate. European Journal of Operational Research, 192(1): 79-92, 2009.

[27] J. T. Teng. et al. Optimal pricing and ordering policy under permissible delay in payments.InternationalJournal of Production Economics, 97: 121-129, 2005.

[28] H. M. Wee. A deterministic lot size inventory model for deteriorating items with shortages and a decliningmarket.Computers and Operations Research, 22: 345-356, 1995.

[29] T. M. Whitin. Theory of inventory management.Princeton University Press, Princeton, NJ, 62-72, 1957.

[30] K. S. Wu. An EOQ inventory model for items with Weibull distribution deterioration, ramp type demandrate and partial backlogging. Production Planning & Control, 12(8): 787-793, 2001.