Some Oscillation Properties of Third Order Linear Neutral Delay Difference Equations

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ABSTRACT

In this paper, we establish some sufficient conditions for the oscillation of solutions of third order linear neutral delay difference equations of the form

\[ \Delta \left( a(n) \Delta^2 (x(n) + p(n)x(\tau(n))) \right) + q(n)x(\sigma(n)) = 0. \]

I. INTRODUCTION

In this paper, we consider the third order linear neutral delay difference equations from

\[ \Delta \left( a(n) \Delta^2 (x(n) + p(n)x(\tau(n))) \right) + q(n)x(\sigma(n)) = 0 \tag{1} \]

where \( n \in N(n_0) = \{n_0, n_0 + 1, \ldots\} \), \( n_0 \) is a nonnegative integer, subject to the following conditions

\( (H_1) \) \( a(n), p(n), q(n) \) are positive sequences.

\( 0 \leq p(n) \leq p \leq 1, \tau(n) \leq n, \sigma(n) \leq n, \lim \tau(n) = \lim \sigma(n) = \infty \) and \( R(n) = \sum_{s=n_0}^{n-1} \frac{1}{a(s)} \to \infty \) as \( n \to \infty \)

\( (H_2) \) \( \sum_{n=n_0}^{\infty} \sum_{s=n_0}^{\infty} \frac{1}{a(s)} \sum_{\sigma(n)=0}^{n} q(t) = \infty. \)

\( (H_3) \) \( \limsup_{n \to \infty} \sum_{s=n_0}^{n-1} q(s)(1 - p(s)) \left( \frac{KM(s^2)}{2} \right) - \frac{a^2(s+1)}{4sa(s)} = \infty. \)

We set \( \zeta(n) = x(n) + p(n)x(\tau(n)). \)

The oscillation theory of difference equations and their applications have received more attention in the last few decades, see [[1]-[4]], and the references cited therein. Especially the study of oscillatory behavior of second order equations of various types occupied a great deal of interest. However the study of third order difference equations have received considerably less attention even though such equations have wide applications. In [[5]-[10]] the authors investigated the oscillatory properties of solutions of third order delay difference equations and in [[11]-[15]]. Motivated by the above observations, in this paper, we investigate the oscillatory behavior of solutions of equation (1).

Let \( \theta = \max \left\{ \lim_{\delta \to 0} \sigma(n), \tau(n) \right\}. \) By a solution of equation (1) we mean a real sequence \( x(n) \) which is defined for all \( n \geq n_0 - \theta \) satisfying (1) for all \( n \geq n_0 \). A non-trivial solution \( x(n) \) is said to be oscillatory if it is neither eventually positive or eventually negative; otherwise, it is nonoscillatory. Equation (1) is said to be oscillatory if all its solutions are oscillatory.

II. MAIN RESULTS

**Lemma 2.1.** Let \( x(n) \) be a positive solution of equation (1) for all \( n \geq n_0 \) such \( x(n) > 0, \Delta x(n) \geq 0, \) and \( \Delta^2 x(n) \leq 0 \) on \( [n_1, \infty) \) for some \( n_1 \geq n_0. \) Then for each \( k \) with \( 0 < k < 1, \) there exists \( n_2 \geq n_1 \) such that

\[ \frac{x(n - \sigma)}{x(n)} \geq k \cdot \frac{n - \sigma}{n}, \quad n \geq n_2. \tag{2} \]

**Proof.** From the Lagrange’s Mean value theorem, we have for \( n \geq n_1, \) for some \( k \)
\[
\Delta x(k) = \frac{x(n) - x(\sigma(n))}{n - \sigma(n)}; \text{ for some } k
\]

(3)
such that \(\sigma(n) < k < n, \Delta^2 x(n) \leq 0\) and \(\Delta x(n)\) is non-increasing, which implies that \(\Delta x(k) < \Delta x(\sigma(n))\) and hence, using equation (3)

\[
x(n) \leq x(\sigma(n)) + \Delta x(\sigma(n))(n - \sigma(n))
\]

(4)

Apply Lagrange’s Mean value theorem once again for \(x(n)\) on \([n_1, \sigma(n)]\) for \(n \geq n_1 + \sigma(n)\). Now

\[
\Delta x(c) = \frac{x(\sigma(n)) - x(n_k)}{\sigma(n) - n_k}
\]

for some \(c\) such that \(n_1 < c < \sigma(n)\) and \(\Delta x(c) > \Delta x(\sigma(n))\) which implies

\[
x(\sigma(n)) \geq \Delta x(\sigma(n))(\sigma(n) - n_1). \text{ Hence}
\]

\[
\frac{x(\sigma(n))}{\Delta x(\sigma(n))} \geq \sigma(n) - n_1
\]

For \(K \in (0,1)\), we can find \(n_2 \geq n_1 + \sigma\)

\[
\frac{x(\sigma(n))}{\Delta x(\sigma(n))} \geq K\sigma(n) \text{ for } n \geq n_2
\]

(5)

From equation (4) and for all \(n \geq n_2\), we have

\[
\frac{x(n)}{x(\sigma(n))} \leq 1 + \frac{1}{K\sigma(n)}(n - \sigma(n))
\]

\[
\leq 1 + \frac{n}{K\sigma(n)} - \frac{\sigma(n)}{K\sigma(n)}
\]

\[
\leq \frac{n}{K(\sigma(n))}
\]

Hence,

\[
\frac{x(\sigma(n))}{x(n)} \geq \frac{K(\sigma(n))}{n}
\]

(6)

**Lemma 2.2** Let \(x(n)\) be a positive solution of equation (1), then the corresponding sequence \(z(n)\) satisfies the following condition \(z(n) > 0, \Delta z(n) > 0, \Delta^2 z(n) > 0, \Delta^3 z(n) > 0\) for some \(n_1 \geq n_0\). Then there exists \(n_2 \geq n_1\) such that \(\frac{z(n)}{\Delta z(n)} \geq \frac{Mn}{2}, \ n \geq n_2\) for each \(M, 0 < M < 1\).

**Proof.** We define a function \(H(n)\) for \(n \geq n_2 \geq n_1\), as

\[
H(n) = (n - n_2)z(n) - \frac{M(n - n_2)^2}{2} \Delta z(n)
\]

(7)

\[
\Delta H(n) \geq z(n) + (n - n_2)\Delta z(n) - \frac{M(n - n_2)^2}{2} \Delta^2 z(n)
\]

(8)

By Taylor’s Theorem,

\[
z(n) \geq z(n_2) + (n - n_2)\Delta z(n_2) + \frac{(n - n_2)^2}{2} \Delta^2 z(n)
\]

From (8)

\[
\Delta H(n) \geq z(n_2) + (n - n_2)\Delta z(n_2) + \frac{(n - n_2)^2}{2} \Delta^2 z(n) + (n - n_2)\Delta z(n) - \frac{M(n - n_2)^2}{2} \Delta^2 z(n)
\]

which implies \(\Delta H(n) > 0\) and \(H(n + 1) > H(n) > H(n_2) = 0\) for every \(n \geq n_2\) from (7)
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\[(n-n_2)z(n) - \frac{M(n-n_2)^2}{2} \Delta z(n) > 0\]

which implies \(\frac{z(n)}{\Delta z(n)} \geq \frac{Mn}{2}\) for \(n \geq n_2\).

**Theorem 2.3.** Assume that \((H_1)\) to \((H_2)\) hold, then equation (1) is oscillatory.

**Proof:** Suppose, if possible that the equation (1) has a nonoscillatory solution. Without loss of generality suppose that \(x(n)\) is a positive solution of equation (1). We shall discuss the following cases for \(z(n)\).

(i) \(z(n) > 0, \Delta z(n) < 0, \Delta^2 z(n) > 0, \Delta^3 z(n) \leq 0,\)

(ii) \(z(n) > 0, \Delta z(n) > 0, \Delta^2 z(n) > 0, \Delta^3 z(n) \leq 0,\)

**Case 1.** \(z(n) > 0, \Delta z(n) < 0, \Delta^2 z(n) > 0, \Delta^3 z(n) \leq 0,\)

Since \(z(n) > 0\) and \(\Delta z(n) < 0\), then there exists finite limits \(\lim_{n\to\infty} z(n) = k\). We shall prove that \(k = 0\).

Assume that \(k > 0\). Then for any \(\varepsilon > 0\), we have \(k + \varepsilon > z(n) > k\). Let \(0 < \varepsilon < \frac{k(1-p)}{p}\), we have \(k + \varepsilon > x(n) > k - p(n)x(\tau(n))\).

\(x(n) > k - p(k + \varepsilon) = m(k + \varepsilon)\)

\(x(n) > mz(n)\)

When \(m = \frac{k - p(k + \varepsilon)}{(k + \varepsilon)}\). Now from the equation (1) we have

\[\Delta \left(a(n)\Delta^2 \left(\left(\tau(n) + p(n)x(\tau(n))\right)\right)\right) = -q(n)x(\sigma(n)) - \Delta \left(a(n)\Delta^2 z(n)\right) \geq q(n)mz(\sigma(s))\]

Summing the above inequality from \(n\) to \(\infty\) we get,

\[\sum_{t=n}^{\infty} \Delta \left(a(t)\Delta^2 z(t)\right) \geq m \sum_{s=n}^{\infty} q(s)z(\sigma(s))\]

\[a(n)\Delta^2 z(n) \geq m \sum_{s=n}^{\infty} q(s)z(\sigma(s))\]

Using the fact that \(z(\sigma(n)) \geq k\) we obtain, \(a(n)\Delta^2 z(n) \geq mk \sum_{s=n}^{\infty} q(s)\) which implies

\[\Delta^2 (z(n)) \geq mk \left(\frac{1}{a(n)} \sum_{s=n}^{\infty} q(s)\right)\]

Summing from \(n\) to \(\infty\) we have,

\[\sum_{s=n}^{\infty} \Delta^2 z(s) \geq mk \sum_{s=n}^{\infty} \left(\frac{1}{a(s)} \sum_{t=s}^{\infty} q(t)\right)\]

\[-\Delta z(n) \geq mk \sum_{s=n}^{\infty} \left(\frac{1}{a(s)} \sum_{t=s}^{\infty} q(t)\right)\]

Summing the last inequality \(n_1\) to \(\infty\)

\[z(n_1) \geq mk \sum_{n=n_1}^{\infty} \sum_{s=n}^{\infty} \left(\frac{1}{a(s)} \sum_{t=s}^{\infty} q(t)\right)\]

This contradicts \((H_2)\). Thus \(k = 0\). Moreover, the inequality \(0 \leq x(n) \leq z(n)\) implies \(\lim_{n\to\infty} x(n) = 0\).

**Case 2.** \(z(n) > 0, \Delta z(n) > 0, \Delta^2 z(n) > 0, \Delta^3 z(n) \leq 0,\)

We have \(x(n) = z(n) - p(n) x(\tau(n))\), we obtain further

\[x(\sigma(n)) = z(\sigma(n)) - p(\sigma(n))x(\sigma(n) - \tau) \geq z(\sigma(n)) - p(\sigma(n))x(\sigma(n))\]
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From equation (1) we have,

$$\Delta \left( a(n)\Delta^2 z(n) \right) \leq -q(n)x(\sigma(n))$$

$$\Delta \left( a(n)\Delta^2 z(n) \right) \leq -q(n)(1 - p(\sigma(n)))z(\sigma(n))$$

$$w(n) = n \frac{a(n)\Delta^2 z(n)}{\Delta z(n)} \geq n \geq n_1$$

$$\Delta w(n) = \left( \frac{a(n+1)\Delta^2 z(n+1)}{\Delta z(n+1)} \right) + n \left( \frac{\Delta(a(n)\Delta^2 z(n))}{\Delta z(n+1)} - \frac{a(n)\Delta^2 z(n)\Delta^2 z(n+1)}{\Delta z(n)\Delta z(n+1)} \right)$$

$$\leq \frac{w(n+1)}{n+1} + n \left( \frac{\Delta(a(n)\Delta^2 z(n))}{\Delta z(n)} - \frac{a(n)(\Delta^2 z(n+1))^2}{(\Delta z(n))^2} \right)$$

$$\leq \frac{w(n+1)}{n+1} - \frac{na(n)(1 - p(\sigma(n)))z(\sigma(n))}{\Delta z(n)} - \frac{na(n)}{(n+1)^2a^2(n+1)}w^2(n+1)$$

(10)

Also from Lemma (2.1) with $$x(n) = \Delta z(n)$$

$$\frac{x(\sigma(n))}{x(n)} \geq \frac{K\sigma(n)}{n}, \sigma(n) \geq n$$

$$\frac{\Delta z(\sigma(n))}{\Delta z(n)} \geq \frac{K\sigma(n)}{n}$$

$$\frac{1}{\Delta z(n)} \geq \frac{K\sigma(n)}{n} \frac{1}{\Delta z(\sigma(n))}$$ for $$\sigma(n) \geq n_1 \geq n_2$$.

By Lemma (2.2)

$$\frac{z(\sigma(n))}{\Delta z(n)} \geq \frac{K\sigma(n)}{n} \frac{z(\sigma(n))}{\Delta z(\sigma(n))}$$

$$\geq \frac{K\sigma(n)}{n} \frac{M\sigma(n)}{2}$$

$$\frac{z(\sigma(n))}{\Delta z(n)} \geq \frac{KM(\sigma(n))^2}{2n}$$

(12)

Using (11) and (12) in (10)

$$\Delta w(n) \leq -q(n)(1 - p(\sigma(n)))\left( \frac{KM\sigma^2(n)}{2} \right) + \frac{w(n+1)}{n+1} - \frac{na(n)}{(n+1)^2a^2(n+1)}w^2(n+1)$$

(13)

Using the inequality

$$Vx - Ux^2 \leq \frac{1}{4}V^2, U > 0$$

And put $$x = \frac{w(n+1)}{n+1}, V = \frac{1}{n+1}, U = \frac{na(n)}{(n+1)^2a^2(n+1)}$$, we have

$$\frac{w(n+1)}{(n+1)} - \frac{na(n)}{(n+1)^2a^2(n+1)}w^2(n+1) \leq \frac{a^2(n+1)}{4na(n)}$$

(14)

From equation (13)
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\[ \Delta w(n) \leq -nq(n)(1 - p(\sigma(n)))\left(\frac{KM(\sigma(n))^2}{2n}\right) + \frac{a^2(n + 1)}{4na(n)} \]  

(15)

Summing the last inequality from \( n_2 \) to \( n - 1 \) we obtain

\[ \sum_{n=n_2}^{n-1} q(s)(1 - p(\sigma(s)))\left(\frac{KM(\sigma(s))^2}{2}\right) - \frac{a^2(s + 1)}{4sa(s)} \leq w(n_2) \]

Taking lim sup in the above inequality, we obtain contradiction with \( H_3 \).

REFERENCES