I. Introduction

Exponential distribution has been used widely in life testing applications. Estimation and testing of the parameters of the two-parameters exponential distribution were addressed by many authors. Hirano (1984) gave a preliminary test procedure for the scale parameters of exponential distribution when selection parameters is unknown. Wright et al (1978) gave inferences for the two-parameters exponential distribution under Type 1 censored sampling. Abu-Salih (2015) derived different tests for the scale parameters of the two-parameters exponential distribution and composed the power and the expected simple size of these tests. Different sampling methods were used in the estimation and testing procedures for this distribution. Median ranked set sampling was used by Hossain and Muttlak(2006). Abu-Dayyeh and Muttlak (1996) used Ranked set Sampling (RSS) for their tests on the scale parameter of the exponential distribution. Shahi (2012) used extreme RSS for hypothesis testing.

Ranked Set sampling (RSS) was originally used by McIntyre(1952) to improve the estimation of population mean of agricultural Grapes. It is mainly used in situations whereby ranking the items is easy while measuring them is expensive. In this research the modified rank set sampling (MRSS) in testing the scale parameter of the two-parameters exponential distribution. Since RSS relies on ranking samples the size of each set must be small for each of ranking and to minimize ranking errors Usually the size is around 3 or 4.

The steps followed to draw a (MRSS) are:
1. Take a simple random sample of size $r^2$ from the population, and allocate them randomly in $r$ sets of size $r$ each.
2. Rank the units in each row without measuring them. The result of this step is given:

   \[(\chi_{11}, \chi_{21}, \ldots, \chi_{r1})\]

3. To get a modified RSS, the first column of the first cycle is taken, namely \((\chi_{11}, \chi_{12}, \ldots, \chi_{1r})\). The MRSS of cycle (C) will be denoted by \((\chi_{11C}, \chi_{12C}, \ldots, \chi_{1rC})\).
4. Repeat steps (1) to (3) $L$ times, which gives a sample of size $n = rL$.

   The MRSS obtained from the $L$ cycles are independent and identically distributed.

Since $\chi_{1i}$, $i=1, \ldots, rL$, is distributed as $\chi_{11}$ i.e., the first order statistic in a random sample of size $m$ from the original distribution. The MRSS, using $L$ cycles, each of size $r^2$ will be:

\[(\chi_{11}, \chi_{12}, \ldots, \chi_{11}, \chi_{12}, \ldots, \chi_{1r}, \chi_{12}, \ldots, \chi_{1r})\]
II. Statement of the problem

As is well known, there are many tests on the scale parameter of the 2-parameter exponential distribution because of the importance of this distribution. In this research, it is proposed to derive other tests and use MRSS instead of simple random sampling (SRS). These tests are expected to have smaller expected sample size than the fixed sample size.

Also, instead of using SRS of size \( n \), MRSS is used for these tests.

Let \( X \) be a random variable having exponential distribution with pdf:

\[
F(x) = \begin{cases} 
 1 - e^{-\lambda x} & \text{for } \lambda > 0, x > 0 \\
 0 & \text{otherwise}
\end{cases}
\]

Where \( \lambda \) and \( \theta \geq 0 \) are the scale and location parameters respectively. Based on the MRSS procedures discussed in the introduction, the tests will be based on a random sample of size \( n = mk \), which is denoted by

\[
(\chi_1^{(1)}, \chi_2^{(1)}, \ldots, \chi_{mk}^{(1)}), \ldots, (\chi_1^{(L)}, \chi_2^{(L)}, \ldots, \chi_{mk}^{(L)})
\]

Where \( \chi_{ij} \) denotes the first order statistic of the \( i \)-th row of the \( j \)-th cycle consisting of \( r \) independent observations and ranked as in step (2)

It is seen that this sample is of size \( n = rL \) usually \( r \) is small to enable good ranking. The tests considered are to test \( H_0: \lambda \geq \lambda_0 \) vs \( H_1: \lambda < \lambda_0 \) when both parameters \( \lambda \) and \( \theta \) are unknown.

The following tests will be considered:

(i) The UMP (uniformly most powerful) invariant size \( \alpha \) test

Let \( U_n = \frac{\sum_{j=1}^{r} \chi_1^{(j)}}{n} \) and \( C_1 \) is determined from \( P(S_1 > C_1 \mid \lambda_0) = \alpha \)

The power function of \( \phi_1 \) is easily found to be

\[
P_{\phi_1}(\lambda) = P(S_1 < C_1 \mid \lambda) = P(2\lambda S_1 > 2\lambda C_1)
\]

Where \( 2\lambda S_1 \) has a \( \chi^2 \) distribution with \( 2(n-1) \) d.f.

(ii) The size test \( \phi_2 \) based on the range.

Let \( M_n = \frac{\sum_{j=1}^{r} \chi_1^{(j)}}{n} \) and \( C_2 \) is determined from \( P(R_n > C_2 \mid \lambda_0) = \alpha \)

To evaluate this probability and solve for \( C_2 \) and evaluate the power function of \( \phi_2 \), the pdf of \( R_n \) is needed.
Test of scale Parameter of the two-parameters exponential distribution

\[ V(u_n, m_n; \lambda, \theta) = n(n-1) \left[ G \left( \frac{u_n}{m_n} \right) - G \left( \frac{1}{m_n} \right) \right]^{n-2} \]

where \( G \) is the cumulative distribution function of \( Y \). 

\[ G(y) = \begin{cases} 0 & ; y < \theta \\ 1 - e^{-r\lambda(y-\theta)} & ; y \geq \theta \end{cases} \]

It is easy to show that the pdf of \( R_n \) is given by

\[ R_n(t) = n(\lambda)(n-1)e^{-\lambda t} \]

The cumulative distribution of \( R_n \) is:

\[ Q(t) = \int_0^t e^{-\lambda y} \left( 1 - e^{-r\lambda y} \right)^{n-2} dy \]

From (2.3), it can be seen that \( R_n \) tends to be small when \( \lambda \) is large and hence it is reasonable to use the test \( \phi_2 \) for which \( C_2 \) is determined from

\[ P(R_n > C_2 \mid \lambda_0) = (1 - e^{-r\lambda_0 C_2})^{n-1} = \alpha \]

And hence:

\[ C_2 = \frac{1}{2} \int_0^\infty \lambda t \left( 1 - \left( 1 - \alpha \right) \right) \left( 1/(n-1) \right) dy \]

By determining \( C_2 \), the power function of the test \( \phi_2 \) is easily determined and is given by

\[ P(\phi_2(\lambda)) = 1 - e^{-r\lambda C_2} \]

where \( C_2 \) is defined in (2.4) and \( n = \]

### III. **Truncated sequential test**

Instead of using data from all cycles in the MRSS process, it is suggested to use cycles one by one, and when the test is significant, the null hypothesis is rejected, the test stops as shown in Fig.1. No decision of rejection is reached by observing the cycle, the 1st cycle is observed and decision is reached.

<table>
<thead>
<tr>
<th>Cycles</th>
<th>1</th>
<th>2…</th>
<th>K…</th>
<th>(k+1)…</th>
<th>L</th>
</tr>
</thead>
<tbody>
<tr>
<td>Observation</td>
<td>r</td>
<td>r</td>
<td>r</td>
<td>……</td>
<td>1</td>
</tr>
</tbody>
</table>

**Fig. 1**

The test is defined as follows:

The \( k^{th} \) cycle is observed Fig. 1. Define

\[ T_k = \sum_{i=1}^k \sum_{i=1}^r (Y_{ij} - U_k) \]

\[ \min_{\text{any}} V_{ij} \]

where \( U_k = 1 \iff i \leq k \leq r \)

\( T_k \) is monotone non-decreasing in \( k \). To prove this fact, consider the two cases:

(i) \( Y_{ikr} \leq U_k \) for \( 1 \leq i \leq r \)

then \( U_{k+1} = Y_{i_0(k+1)} \) for some \( i_0, 1 \leq i_0 \leq r \)

and
\[
T_{k+1} = \sum_{j=1}^{k+1} \sum_{r=1}^{r} (Y_{ij} - U_{k+1}) = \sum_{j=1}^{k+1} \sum_{r=1}^{r} (Y_{ij} - Y_{(k+1)}) = \sum_{j=1}^{k+1} \sum_{r=1}^{r} (Y_{ij} - Y_{(k+1)})
\]

\[
= \sum_{j=1}^{k+1} \min_{1 \leq i \leq r} Y_{i(k+1)} - U_k
\]

\[
E_{k+1} = T_{k+1} = T_k + \sum_{j=1}^{k} \sum_{r=1}^{r} (Y_{ij} - U_k)
\]

Since \( T_k \) is monotone non-decreasing in \( k \), it is useful to use the truncated sequential test based on \( T_k \), t test \( H_0 : \lambda \geq \lambda_0 \) vs \( H_1 : \lambda < \lambda_0 \) with the possibility that this test will lead to early rejection of \( H_0 \).

Let sampling procedure be the MRSS with numbers of items in each cycle \( r^2 \) and the number of measured observations \( r \) as explained in the setup explained earlier. Let the number of cycles be \( L \), where \( L = n \), the total number of observations.

Consider the test \( \phi_3 \) which rejects \( H_0 : \lambda \geq \lambda_0 \) at the \( k \)th stage of sampling (the \( k \)th cycle), where \( 2 \leq k \leq L-1 \), if \( T_k > C \) and takes one more observation if \( T_k < C \) where \( C \) is given by \( P(T_L > C \mid \lambda_0) = \alpha \).

Notice that \( T_L = S_1 \) and therefore \( C = C_1 \) which was determined before.

If the rule dose not reject \( H_0 \) by using the \((l-1)\)th cycle, then the \( L \)th cycle is used and a decision is reached.

\[ (3.1) \quad \text{Power of test } \phi_3 \]

Let \( E_k = (T_k > C) \) for \( k = 2, 3, \ldots, L \)

Then \( E_2 \subset E_3 \subset \ldots \subset E_L \) since \( T_k \) is monotone non-decreasing.

Hence, \( P(\text{rejecting } H_0) = P(U_{1 \leq i \leq 2}) = P(E_L) \).

And hence the power of the test \( \phi_3 \) is equal to the power of the fixed sample size test \( \phi_1 \) defined by:

\[ \text{Reject } H_0 \text{ if } S_1 > C_1. \]

\[ (3.2) \quad \text{Expected sample size of } \phi_3 \]

Let \( W \) be the number of cycles required for termination of the test in the early rejection procedure.

Since \( E_2 \subset E_3 \subset \ldots \subset E_L \), then

\[ E_k = U_1(R = 2)^{K(L - 1)} \bigcup \left( \left\{ \left( E_i \right) \cap \left( \lambda + 1 \right) \right\} \bigcap E_2 \right) \]

And hence:

\[ (3.3) E(W \mid \lambda) = 2P(E_2 \mid \lambda) + \sum_1 (R = 2)^{K(L - 1)} P(\left( \lambda + 1 \right) \cap E_i \mid \lambda) \]

\[ nP(E_B \mid \lambda) \]

Following the same procedure given by Abu-Salih (2015), \( E(W \mid \lambda) \) is evaluated to be given by:

\[ E(W \mid \lambda) = 2[1 - P(\chi^2_{2r} < 2r\lambda C)] \]
Test of scale Parameter of the two-parameters exponential…

\[ \sum_{k=1}^{n} \left( \frac{(k+1)^2}{\lambda k!} \right) e^{-\lambda t} \lambda^k = n \Phi \chi^2_{(n-1)} < 2n \lambda C. \]

Where \( n = t \)

IV. Comparison of power function of the tests

Power function of tests \( \Phi_1, \Phi_2, \) and \( \Phi_3 \) are computed for \( \lambda_o = 0.4 \), 
\( \ldots, 0.04 \), \( r = 5 \), and \( \lambda = 2, 4, \) and \( 6 \).

Note that power of \( \Phi_3 \) is not listed because it is equal to power of \( \Phi_1 \).

Table 4.1 : powers of tests according to \( \lambda \) and \( t \) when \( r = 5 \)

<table>
<thead>
<tr>
<th>( \lambda )</th>
<th>0.40</th>
<th>0.32</th>
<th>0.24</th>
<th>0.16</th>
<th>0.08</th>
<th>0.04</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r )</td>
<td>2</td>
<td>1.6</td>
<td>1.2</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
</tr>
<tr>
<td>( t )</td>
<td>0.05</td>
<td>0.134939</td>
<td>0.3741097</td>
<td>0.896541</td>
<td>0.989996</td>
<td>0.999715</td>
</tr>
</tbody>
</table>

It is clear that the values of the power of \( \Phi_1 \) and the corresponding truncated sequential test \( \Phi_3 \) are greater than the power of \( \Phi_2 \).

Also, the powers increase with \( t \) and the distance of \( \lambda \) from \( \lambda_o \).

Expected value of number of cycles for \( \Phi_3 \), when \( r = 3 \) and for values of \( \lambda_o = 0.6 \) and values of \( \lambda = 0.60 \) to 0.07 and \( \lambda = 5 \) and 10 are given in Table 2.

Table 2 : E[k | \( \lambda \)] for test \( \Phi_3 \)

<table>
<thead>
<tr>
<th>( \lambda )</th>
<th>0.6</th>
<th>0.53</th>
<th>0.40</th>
<th>0.33</th>
<th>0.20</th>
<th>0.07</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r )</td>
<td>1.8</td>
<td>1.6</td>
<td>1.2</td>
<td>1.0</td>
<td>0.6</td>
<td>0.2</td>
</tr>
<tr>
<td>( \lambda_o )</td>
<td>2.9699</td>
<td>2.9231</td>
<td>2.6597</td>
<td>2.38734</td>
<td>1.638552</td>
<td>0.81337</td>
</tr>
</tbody>
</table>

It is clear that \( E[k | \lambda] < L \), meaning that the truncated sequential test of \( \lambda \) using MRSS is expected to reach decision earlier than the fixed sample size test using MRSS, with \( n = rL \).

V. Conclusion

It has been suggested to use these tests and modified rank set sampling to test \( H_0 : \lambda \geq \lambda_o \) vs \( H_1 : \lambda < \lambda_o \)

where the distribution of the population is the negative exponential with location and scale parameters \( \theta \) and \( \lambda \), respectively.

The tests are free of the location parameter \( \theta \) and the proposed truncated sequential test leads to earlier decision where by it uses a number of cycles smaller than the assumed number \( t \).

References