On The Derivatives and Partial Derivatives of A Certain Generalized Hyper geometric Function

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ABSTRACT
In this paper, methods involving derivatives and the Mellin transformation are employed in obtaining finite summations for the $H$-function of two variables and certain special partial derivatives for the $H$-function of two variables with respect to parameters.

KEY WORDS: Derivatives, Partial derivatives, $H$-function of two variables, Mellin transformation.
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I. INTRODUCTION

The $H$-function of two variables defined and represented by Singh and Mandia [12] in the following manner:

\[
H [x, y] = H \left[ y, x \right] = H \left[ a_1, a_2, \ldots, a_n; b_1, b_2, \ldots, b_m; c_1, c_2, \ldots, c_p; d_1, d_2, \ldots, d_q; e_1, e_2, \ldots, e_r; f_1, f_2, \ldots, f_s; x, y; \xi, \eta \right]
\]

\[
= \frac{1}{4\pi} \int_{\xi_1}^{\xi_2} \int_{\eta_1}^{\eta_2} \phi_1 (\xi, \eta) \phi_2 (\xi, \eta) x^\xi y^\eta d\xi d\eta
\]  

(1.1)

Where

\[
\phi_1 (\xi, \eta) = \prod_{j=1}^{n_1} \Gamma \left( 1 - a_j + \alpha_j \xi + A \eta \right) \prod_{j=n_1 + 1}^{n_2} \Gamma \left( a_j - \alpha_j \xi - A \eta \right) \prod_{j=1}^{q_1} \Gamma \left( 1 - b_j + \beta_j \xi + B \eta \right) \prod_{j=q_1 + 1}^{q_2} \Gamma \left( b_j - \beta_j \xi - B \eta \right)
\]

(1.2)

\[
\phi_2 (\xi) = \prod_{j=1}^{n_1} \Gamma \left( 1 - c_j + \gamma_j \xi \right) \prod_{j=n_1 + 1}^{n_2} \Gamma \left( c_j - \gamma_j \xi \right) \prod_{j=1}^{q_1} \Gamma \left( 1 - d_j + \delta_j \xi \right) \prod_{j=q_1 + 1}^{q_2} \Gamma \left( d_j - \delta_j \xi \right)
\]

(1.3)

\[
\phi_3 (\eta) = \prod_{j=1}^{n_1} \Gamma \left( 1 - e_j + E \eta \right) \prod_{j=n_1 + 1}^{n_2} \Gamma \left( e_j - E \eta \right) \prod_{j=1}^{q_1} \Gamma \left( 1 - f_j + F \eta \right) \prod_{j=q_1 + 1}^{q_2} \Gamma \left( f_j - F \eta \right)
\]

(1.4)
Where \( x \) and \( y \) are not equal to zero (real or complex), and an empty product is interpreted as unity
\[ p_i, q_i, n_i, m_j \] are non-negative integers such that \( 0 \leq n_i \leq p_i, 0 \leq m_j \leq q_j \) \((i = 1, 2, 3; j = 2, 3)\). All the
\[ a_j (j = 1, 2, ..., p_i), b_j (j = 1, 2, ..., q_k), c_j (j = 1, 2, ..., p_j), d_j (j = 1, 2, ..., q_k), e_j (j = 1, 2, ..., p_k) \]
parameters, \( \gamma_j \geq 0 (j = 1, 2, ..., p_j), \delta_j \geq 0 (j = 1, 2, ..., q_k) \) (not all zero simultaneously), similarly
\( E_j \geq 0 (j = 1, 2, ..., p_k), F_j \geq 0 (j = 1, 2, ..., q_k) \) (not all zero simultaneously). The exponents
\( K_j (j = 1, 2, ..., n_j), L_j (j = m_j + 1, ..., q_k), R_j (j = 1, 2, ..., n_j), S_j (j = m_j + 1, ..., q_k) \) can take on non-negative values.

The contour \( L_1 \) is in \( \zeta \) -plane and runs from \(-i\infty \) to \(+i\infty \). The poles of \( \Gamma \left( d_j - \delta_j \xi \right) \) \((j = 1, 2, ..., m_j) \) lie to
the right and the poles of \( \Gamma \left( \left( 1 - c_j + \gamma_j \xi \right) ^k_j \right) \) \((j = 1, 2, ..., n_j) \) \((\text{not all zero simultaneously})\) to
the left of the contour. For \( K_j (j = 1, 2, ..., n_j) \) not an integer, the poles of gamma functions of the numerator
in (1.3) are converted to the branch points.

The contour \( L_2 \) is in \( \eta \) -plane and runs from \(-i\infty \) to \(+i\infty \). The poles of \( \Gamma \left( f_j - F_j \eta \right) \) \((j = 1, 2, ..., m_j) \) lie to
the right and the poles of \( \Gamma \left( \left( 1 - e_j + E_j \eta \right) ^k_j \right) \) \((j = 1, 2, ..., n_j) \) \((\text{not all zero simultaneously})\) to
the left of the contour. For \( R_j (j = 1, 2, ..., n_j) \) not an integer, the poles of gamma functions of the numerator
in (1.4) are converted to the branch points.

The functions defined in (1.1) is an analytic function of \( x \) and \( y \), if
\[
U = \sum_{j=1}^{p_i} a_j + \sum_{j=1}^{p_k} \gamma_j - \sum_{j=1}^{q_k} \beta_j - \sum_{j=1}^{q_k} \delta_j < 0 \tag{1.5}
\]
\[
V = \sum_{j=1}^{p_i} A_j + \sum_{j=1}^{p_k} E_j - \sum_{j=1}^{q_k} B_j - \sum_{j=1}^{q_k} F_j < 0 \tag{1.6}
\]

The integral in (1.1) converges under the following set of conditions:
\[
\Omega = \sum_{j=1}^{n_i} a_j - \sum_{j=n_i+1}^{n_i} \alpha_j + \sum_{j=1}^{n_k} \delta_j - \sum_{j=n_k+1}^{n_k} \beta_j + \sum_{j=1}^{n_k} K_j - \sum_{j=n_k+1}^{n_k} \gamma_j - \sum_{j=1}^{n_k} \beta_j > 0 \tag{1.7}
\]
\[
\Lambda = \sum_{j=1}^{n_i} A_j - \sum_{j=n_i+1}^{n_i} A_j + \sum_{j=1}^{n_k} F_j - \sum_{j=n_k+1}^{n_k} F_j - \sum_{j=1}^{n_k} E_j - \sum_{j=n_k+1}^{n_k} B_j > 0 \tag{1.8}
\]
\[
|\arg x| < \frac{1}{2} \Omega \pi, |\arg y| < \frac{1}{2} \Lambda \pi \tag{1.9}
\]

The behavior of the \( \overline{H} \) -function of two variables for small values of \(|z|\) follows as:
\[ H(x, y) = 0 \left( |x|^2 \right), \max \{ |x|, |y| \} \rightarrow 0 \quad (1.10) \]

Where

\[ \alpha = \min_{1 \leq j \leq m} \left[ \text{Re} \left( \frac{d_j}{\beta_j} \right) \right], \quad \beta = \min_{1 \leq j \leq m} \left[ \text{Re} \left( \frac{f_j}{E_j} \right) \right] \quad (1.11) \]

For large value of \(|z|\),

\[ H(x, y) = 0 \left( |x|^2, |y|^2 \right), \min \{ |x|, |y| \} \rightarrow 0 \quad (1.12) \]

where

\[ \alpha^* = \max_{1 \leq j \leq m} \text{Re} \left( \frac{c_j - 1}{y_j} \right), \quad \beta^* = \max_{1 \leq j \leq m} \text{Re} \left( \frac{e_j - 1}{E_j} \right) \quad (1.13) \]

Provided that \( U < 0 \) and \( V < 0 \).

If we take \( K_j = 1(j = 1, 2, \ldots, n_2), L_j = 1(j = m_2 + 1, \ldots, q_2) \), \( R_j = 1(j = 1, 2, \ldots, m_2) \), \( S_j = 1(j = m_2 + 1, \ldots, q_2) \) in (2.1), the \( H \)-function of two variables reduces to \( H \)-function of two variables due to [9].

If we set \( n_1 = p_1 = q_1 = 0 \), the \( H \)-function of two variables breaks up into a product of two \( H \)-function of one variable namely

\[ H_{p_2, q_2} = \prod_{j=1}^{n_2} \frac{c_j - 1}{y_j} \left( \frac{\beta_j}{\alpha_j} \right) \quad (1.14) \]

If \( \lambda > 0 \), we then obtain

\[ \lambda^2 H_{p_2, q_2} = \prod_{j=1}^{n_2} \frac{c_j - 1}{y_j} \left( \frac{\beta_j}{\alpha_j} \right) \quad (1.15) \]

\[ \lambda^2 H_{p_2, q_2} = \prod_{j=1}^{n_2} \frac{c_j - 1}{y_j} \left( \frac{\beta_j}{\alpha_j} \right) \quad (1.16) \]
II. MAIN RESULTS

If \( t \) be an arbitrary parameter and \( \alpha, \alpha' \) be positive real numbers, then it can be verified that

\[
D^\prime \frac{H}{t^\alpha} \left[ z^{t^\alpha}, z^{t^{\alpha'}} \right] = t^{-1-\alpha} H \left[ z^{t^\alpha} + z^{t^{\alpha'}} \right] \quad (2.1)
\]

And

\[
t^{-1-\alpha} H \left[ z^{t^{\alpha}} + z^{t^{\alpha'}} \right] \bigg| \bigg| \begin{pmatrix} \alpha \\ \alpha' \end{pmatrix} = t^{-1-\alpha} H \left[ z^{t^{\alpha}} \right] + t^{-1-\alpha} H \left[ z^{t^{\alpha'}} \right] = 0
\]

Differentiating (2.2) two times w.r.t. \( t \) and simplifying, it follows by induction that

\[
H \left[ z^{t^{\alpha}} + z^{t^{\alpha'}} \right] \bigg| \bigg| \begin{pmatrix} \alpha \\ \alpha' \end{pmatrix} = \sum_{n_1, n_2 = 0}^{\infty} \binom{n_1 + n_2}{n_1} \binom{n_1 + n_2 + 1}{n_1 + 1} H \left[ z^{t^{\alpha}} \right] + \binom{n_1 + n_2 + 1}{n_1 + 1} H \left[ z^{t^{\alpha'}} \right] \quad (2.3)
\]

(2.3) readily admits an extension and we have

\[
= \sum_{n_1, n_2 = 0}^{\infty} \binom{n_1 + n_2}{n_1} \binom{n_1 + n_2 + 1}{n_1 + 1} H \left[ z^{t^{\alpha}} \right] + \binom{n_1 + n_2 + 1}{n_1 + 1} H \left[ z^{t^{\alpha'}} \right] \quad (2.4)
\]

Considering various other forms that (2.1) admits, similar other results can be obtained.

In the next place, in view of (2.1) we note that the 2-dimensional Mellin-transformation \((6),11.2\) \( M^\prime \) of the \( H \) -function of two variables is given by

\[
M^\prime (H) = Q (-\xi, -\eta)
\]

Provided

\[
-\min_{1 \leq j \leq n} \Re \left( \frac{d_j}{\delta_j} \right) < \xi < \max_{1 \leq j \leq n} \Re \left( \frac{c_j - 1}{\gamma_j} \right)
\]
\[ - \min_{1 \leq j < m} \Re \left( \frac{f_j}{F_j} \right) < \eta < \max_{1 \leq j < m} \Re \left( R_j \cdot \frac{e_j - 1}{E_j} \right) \]

We also note that, since (1.7 (30) of Erdelyi [7]) for a positive integer \( N \),

\[ \psi (a + N) - \psi (a) = \sum_{k=1}^{N} \frac{(-1)^{k-1} N!}{k (N-k)!} \psi (a + k) \]

\[ \psi (a + k) = \frac{\Gamma (a + k)}{\Gamma (a)} \]

Partial differentiation of the gamma product \( \Gamma \left( 1 - \frac{e}{2} + \alpha \cdot \xi + \alpha \cdot \eta \right) \Gamma \left( 1 + \frac{e}{2} + \alpha \cdot \xi + \alpha \cdot \eta \right) \) w.r.t. the arbitrary parameter \( e \) at \( e = N \) can be expressed as a finite sum

\[ \frac{\partial}{\partial e} \left\{ \Gamma \left( 1 - \frac{e}{2} + \alpha \cdot \xi + \alpha \cdot \eta \right) \Gamma \left( 1 + \frac{e}{2} + \alpha \cdot \xi + \alpha \cdot \eta \right) \right\} \bigg|_{e=N} \]

\[ = \frac{1}{2} \Gamma \left( 1 - \frac{N}{2} + \alpha \cdot \xi + \alpha \cdot \eta \right) \sum_{k=1}^{N} \frac{(-1)^{k-1} N!}{k (N-k)!} \Gamma \left( 1 + \frac{N}{2} + \alpha \cdot \xi + \alpha \cdot \eta \right) \]

Where \( \alpha \cdot \xi, \alpha \cdot \eta \) are positive real numbers.

Thus for \( n > 0, N > 0 \), we have

\[ M^{\alpha} \{ \frac{N!}{2} \sum_{k=1}^{N} \frac{(-1)^{k-1} N!}{k (N-k)!} \} \]

\[ \times Q (-\xi, -\eta) \]

\[ M^{\alpha} \left\{ \frac{N!}{2} \sum_{k=1}^{N} \frac{(-1)^{k-1} N!}{k (N-k)!} H_{p_1+1, q_1, r_1}^{\alpha, \beta, \gamma, \delta, \epsilon} \right\} \]

\[ = \frac{N!}{2} \sum_{k=1}^{N} \frac{(-1)^{k-1} N!}{k (N-k)!} \]

\[ \times Q (-\xi, -\eta) \]

\[ \frac{N!}{2} \sum_{k=1}^{N} \frac{(-1)^{k-1} N!}{k (N-k)!} \]

\[ \times Q (-\xi, -\eta) \]

(2.5)

But for \( z(t) = u^{-z(t)} \), \( i = 1, 2 \), (2.5) can be written as
\[
\frac{\partial}{\partial e} \left[ H_{p_1 + q_1 + r_1 + s_1} \right]_{\alpha_1} = \sum_{\sigma \in \Sigma} \left( \frac{1}{2} \sum_{i} \left[ \left( \frac{e^{2 \sigma \cdot \omega \cdot r}}{e^{2 \sigma \cdot \omega \cdot r}} \right) \left( \frac{e^{2 \sigma \cdot \omega \cdot r}}{e^{2 \sigma \cdot \omega \cdot r}} \right) \right]_{\alpha_i} \right)
\]
\[
= \frac{N!}{2} \sum_{k=1}^{N} \frac{(-1)^{N} N}{u^{N/2}} \left[ H_{p_1 + q_1 + r_1 + s_1} \right]_{\alpha_1}
\]
\[
D_u^{N-k} \left[ \frac{u^{N/2}}{H_{p_1 + q_1 + r_1 + s_1}} \right]_{\alpha_1}
\]

If we express the derivative into a sum, carry out the differentiations, interchange the order of summation and simplify, we obtain

\[
\frac{\partial}{\partial e} \left[ H_{p_1 + q_1 + r_1 + s_1} \right]_{\alpha_1}
\]
\[
= \frac{N!}{2} \sum_{p=0}^{N} \frac{(-1)^{p}}{p!(N-p)!} \left[ H_{p_1 + q_1 + r_1 + s_1} \right]_{\alpha_1}
\]

Similar other results can be obtained by considering products or quotients of such gamma functions whose partial derivatives w.r.t. the arbitrary parameter involved can be expressed as a finite sum.

For example, for the quotient

\[
\frac{\Gamma(1-e-N+\alpha \cdot \zeta + \alpha \cdot \eta)}{\Gamma(1-e + \alpha \cdot \zeta + \alpha \cdot \eta)}
\]

We have

\[
\frac{\partial}{\partial e} \left[ H_{p_1 + q_1 + r_1 + s_1} \right]_{\alpha_1}
\]
On The Derivatives and Partial Derivatives...

\[ N! \sum_{k=0}^{N} \frac{(-1)^{k-1}}{p(N-k)!} \frac{H_{\alpha_1, \alpha_2; \alpha_1, \alpha_2; \alpha_1, \alpha_2; \alpha_1, \alpha_2}{}_{p+q, q+1}^{r_1, r_2, r_3, r_4, r_5, r_6; p+q, q+1} \left[ \begin{array}{c} x \cr \beta_1, \beta_2, \beta_3 \end{array} \right]_{\alpha_1, \alpha_2; \alpha_1, \alpha_2; \alpha_1, \alpha_2; \alpha_1, \alpha_2} \right] \]

(2.8)

REFERENCES