Lattice Points On The Homogeneous Cubic Equation With Four Unknowns

$$x^2 - xy + y^2 + 3w^2 = 7z^3$$

M.A.Gopalan[1], V.Sangeetha[2] , Manju Somanath[3]

[1] *Professor, Department of Mathematics, Srimathi Indhira Gandhi College,Trichy-*
[2] *Assistant Professor, Department of Mathematics, National College,Trichy-1.*
3Assistant Professor, Department of Mathematics, National College,Trichy-1.

ABSTRACT

The homogeneous cubic equation with four unknowns represented by $x^2 - xy + y^2 + 3w^2 = 7z^3$ is analyzed for its patterns of non zero distinct integral solutions.Three different patterns of solutions are presented. A few interesting relations between the solutions and special numbers ,namely, polygonal numbers,pyramidal numbers,centered pyramidal numbers,star numbers, nasty numbers,dodecahedral number,rhombic dodecahedral number and prism numbers are exhibited.

KEY WORDS: *Homogeneous cubic,Lattice Points,Integral solutions. MSC2000 Mathmatics Subject Classification Number 11D25.*

Notations

$t_{m,n}$= Polygonal number of rank n with sides m.

P_n^m = Three dimensional figurate number of rank n with sides m

cP_n^m = Three dimensional centered figurate number of rank n with sides m

S_n= Star number

$D(n)$ = Dodecahedral number of rank n

$RD(n)$ = Rhombic dodecahedral number of rank n

PCS_m^n = Prism number of rank n with sides m.

I. INTRODUCTION

The Cubic Equation offers an unlimited field for research because of their variety [1,2]. In particular, one may refer [3-13] for cubic equation with four unknowns. This communication concerns with yet another interesting equation $x^2 - xy + y^2 + 3w^2 = 7z^3$ representing a homogeneous cubic with four unknowns for determining its infinitely many non-zero integral solutions. Also a few interesting relations among the solutions have been presented.

II. Method of Analysis

The homogeneous cubic represented by the cubic equation is

$$x^2 - xy + y^2 + 3w^2 = 7z^3 \quad (1)$$

It is observed that (1) is satisfied by infinitely many non-zero distinct integral solutions. For the sake of clear understanding, we present below different patterns of solutions to (1).

2.1 Pattern - 1

Introducing the linear transformations

$$x = u + v\ ;\ y = u - v\ ;\ w = v \quad (2)$$

in (1), it is written as

$$u^2 + 6v^2 = 7z^3 \quad (3)$$

Assume $z = a^2 + 6b^2$ (4)

Write 7 as 7= $(1 + i\sqrt{6})(1 - i\sqrt{6})$ (5)

Using (4) and (5) in (3) and applying the method of factorization, define

$$u + i\sqrt{6}v = (a + i\sqrt{6}b)^3(1 + i\sqrt{6})$$

Equating the real and imaginary parts, we get

$$\left.\begin{aligned} u &= a^3 - 18ab^2 - 18a^2b + 36b^3 \\ v &= a^3 - 18ab^2 + 3a^2b - 6b^3 \end{aligned}\right\} \qquad (6)$$

Using (6) in (2), the integral solutions to (1) is obtained as

$$x(a,b) = 2a^3 - 36ab^2 - 15a^2b + 30b^3$$
$$y(a,b) = -21a^2b + 42b^3$$
$$z(a,b) = a^2 + 6b^2$$
$$w(a,b) = a^3 - 18ab^2 + 3a^2b - 6b^3$$

A few properties of the above solutions are

1. $2x(a(a+1),1) - 6P^6_{a(a+1)} + 66t_{3,a(a+1)} + 76t_{3,a} = 60$.
2. . $x(1,b) - 6P^{30}_b - 18t_{3,b} + 45t_{4,b} = 2$.
3. $y(1,b) - x(1,b) - 6P^{14}_b - 6t_{3,b} - 30t_{4,b} = -2$.
4. $y(1,b) + z(1,b) - 6P^{44}_b - 6t_{3,b} \equiv 1 \pmod{15}$.
5. $x(1,b) + w(1,b) - 6P^{24}_b - 12t_{3,b} + 60t_{4,b} = 3$.

In addition to (5), 7 can also be written as

$$7 = \frac{(13+i\sqrt{6})(13-i\sqrt{6})}{25} \quad \text{and} \quad 7 = \frac{(29+i\sqrt{6})(29-i\sqrt{6})}{121} \qquad (7)$$

Applying the procedure similar to pattern 1, we obtain the corresponding integer solutions to (1) which are given below respectively .

$$x(A,B) = 350A^3 - 6300AB^2 + 525A^2B - 1050B^3$$
$$y(A,B) = 300A^3 - 5400AB^2 - 1425A^2B + 2850B^3$$
$$z(A,B) = 25A^2 + 150B^2$$
$$w(A,B) = 25A^3 - 450AB^2 + 975A^2B - 1950B^3$$

and

$$x(A,B) = 3630A^3 - 65340AB^2 + 8349A^2B - 16698B^3$$
$$y(A,B) = 3388A^3 - 60984AB^2 - 12705A^2B + 25410B^3$$
$$z(A,B) = 121A^2 + 726B^2$$
$$w(A,B) = 121A^3 - 2178AB^2 + 10527A^2B - 21054B^3$$

2.2 Pattern - 2

Introducing the linear transformations

$$x = u + v\ ;\ y = u - v\ ;\ w = u \qquad (8)$$

in (1), it is written as

$$4u^2 + 3v^2 = 7z^3 \qquad (9)$$

Assume $z = 4a^2 + 3b^2$ (10)

Write 7 as $7 = \frac{(5+i\sqrt{3})(5-i\sqrt{3})}{4}$ (11)

Using (10) and (11) in (9) and applying the method of factorization, define

$$2u + i\sqrt{3}v = (2a + i\sqrt{3}b)^3 \frac{5+i\sqrt{3}}{2}$$

Equating the real and imaginary parts, we get

$$u = 10a^3 - \frac{45}{2}ab^2 - 9a^2b + \frac{9}{4} \quad \text{and} \quad v = 4a^3 - 9ab^2 + 30a^2b - \frac{15}{2}b^3$$

As our interest centers on finding integer solution we choose suitable values for *u* and *v*.

Letting $a = A$, $b = 2B$, we get

$$\left.\begin{aligned} u &= 10A^3 - 90AB^2 - 18A^2B + 18B^3 \\ v &= 4A^3 - 36AB^2 + 60A^2B - 60B^3 \end{aligned}\right\} \qquad (12)$$

Using (12) in (8), the integral solutions to (1) is obtained as

$$x(A,B) = 14A^3 - 126AB^2 + 42A^2B - 42B^3$$
$$y(A,B) = 6A^3 - 54AB^2 - 78A^2B + 78B^3$$
$$z(A,B) = 4A^2 + 12B^2$$
$$w(A,B) = 10A^3 - 90AB^2 - 18A^2B + 18B^3$$

Some interesting properties of the above solutions are

1. $2z(1,B) - 8$ is a Nasty Number.
2. $w(1,B) - 6P^{20}_B + 6t_{3,b} + 90t_{4,b} = 10$.

3. $z(A,1)+w(A,1)-6P_A^{12}+34t_{3,A}-t_{4,2A-1}+9t_{4,A}=-91.$
4. $w(A(A+1),1)-2D\big(A(A+1)\big)+6P_{A(A+1)}^5+192t_{3,A}-19$ is a Rhombic dodecahedral number.
5. $x(A,1)-y(A,1)-3CP_A^{16}-240t_{3,A}\equiv 67\ (mod\ 187).$
6. $y(1,B)-2PCS_B^{18}+144t_{3,B}\equiv 0\ (mod\ 2).$

In addition,by writing 7 as 7=$(2+i\sqrt{3})(2-i\sqrt{3})$,we obtain the integer solutions to (1) as below

$$x(A,B)=16A^3-144AB^2+12A^2B-12B^3$$
$$y(A,B)=84B^3-84A^2B$$
$$z(A,B)=4A^2+12B^2$$
$$w(A,B)=8A^3-72AB^2-36A^2B+36B^3$$

III. CONCLUSION

To conclude, one may search for other patterns of solutions and their corresponding properties.

REFERENCES

[1] L.E.Dickson, *History of Theory of Numbers*, Vol.2, Chelsea Publishing Company,New York, 1952.

[2] L.J.Mordell, *Diophantine Equations*, Academic Press, London, 1969.

[3] M.A.Gopalan, S.Premalatha, *Integral Solutions of* $(x+y)(xy+w^2)=2(k^2+1)z^3$, Bulletin of Pure and Applied Sciences, (2009) Vol.29E, No:2, 197-202.

[4] M.A.Gopalan, V.Pandichelvi, *Remarkable Solutions on the cubic equation with four unknowns* $x^3+y^3+z^3=28(x+y+z)w^2$, Antarctica J.of Math,(2010) Vol.7,No.4 ,393-401.

[5] M.A.Gopalan,B.Sivagami, *Integral Solutions of homogeneous cubic equation with four unknowns* $x^3+y^3+z^3=3xyz+2(x+y)w^3$,Impact J.Sci.Tech., (2010) Vol.4,No.3, 53-60.

[6] M.A.Gopalan,S.Premalatha *On the cubic Diophantine Equation with four unknowns* $(x-y)(xy-w^2)=2(n^2+2n)z^3$, International Journal of Mathematical Sciences, (2010),Vol.9, No.1-2, 171-175.

[7] M.A.Gopalan,J.Kaliga Rani, *Integral solutions of* $x^3+y^3+(x+y)xy=z^3+w^3+(z+w)zw$,Bulletin of Pure and Applied Sciences, (2010),Vol.29E,No:1, 169-173.

[8] M.A.Gopalan, S.Premalatha, *Integral Solutions of* $(x+y)(xy+w^2)=2(k+1)z^3$, The Global Journal of Applied Mathematics and Mathematical Sciences, (2010),Vol.3, No.1-2,51-55.

[9] M.A.Gopalan,S.Vidhyalakshmi and S.Mallika,*Observation on cubic equation with four unknowns* $xy+2z^2=w^3$,The Global Journal of Mathematics and Mathematical Sciences,(2012) Vol.2,No:1-2, 69-74.

[10] M.A.Gopalan,S.Vidhyalakshmi and S.Mallika,*Observation on cubic equation with four unknowns* $2(x^3+y^3)=z^3+w^2(x+y)$, IJAMP,(2012) Vol.4,No:2 July-December ,103-107.

[11] M.A.Gopalan,S.Vidhyalakshmi and G.Sumathi,*On the homogeneous cubic equation with four unknowns* $x^3+y^3=14z^3-3w^2(x+y)$,Discovery(2012) Vol.2,No.4, October ,17-19

[12] M.A.Gopalan and K.Geetha, ,*Observation on cubic equation with four unknowns* $x^3+y^3+xy(x+y)=z^3+2(x+y)w^2$, International Journal of Pure and Applied Mathematical Sciences, (2013),Vol.6, No.1, 25-30.

[13] M.A.Gopalan,V.Sangeetha and Manju Somanath,*Lattice Points on the Homogeneous cubic equation with four unknowns* $(x+y)(xy+w^2)=(k^2-1)z^3, k>1$, International Journal of Pure and Applied Mathematical Sciences,(2013),Vol.6,No.2,241-246.