On A Locally Finite In Ditopological Texture Space

1. I. Arockia Rani, 2. A. A. Nithya

ABSTRACT

The present study deals with the new concept namely α-paracompactness in ditopological texture spaces. Also we develop comprehensive theorems using paracompactness and α-open sets. Many effective characterizations and properties of this newly developed concept are obtained.

Keywords: Texture spaces, Ditopology, Ditopological Texture spaces, α-paracompactness, α-locally finite, α-locally co-finite. 2000 AMS Subject Classification. 54C08, 54A20

I. INTRODUCTION

In 1998 L.M. Brown introduced an attractive concept namely Textures in ditopological setting for the study of fuzzy sets in 1998. A systematic development of this texture in ditopology has been extensively made by many researchers [3,4,5,7]. The present study aims at discussing the effect of α-paracompactness in Ditopological Texture spaces. Let S be a set, a texturing T of S is a subset of P(S). If

\( (T, \subset) \) is a complete lattice containing S and φ, and the meet and join operations in \( (T, \subset) \) are related with the intersection and union operations in \( (P(S), \subset) \) by the equalities

\[ \bigcap_{i \in I} A_i = A \cap \bigcap_{i \in I} A_i, \quad A_i \in T, \quad i \in I, \quad \text{for all index sets } I, \]

\[ \bigcup_{i \in I} A_i = \bigcup_{i \in I} A_i, \quad A_i \in T, \quad i \in I, \quad \text{for all finite index sets } I. \]

(2) T is completely distributive.

(3) T separates the points of S. That is, given \( s_1 \neq s_2 \) in S we have \( A \in T \) with \( s_1 \in A, \quad s_2 \notin A, \quad \text{or } A \notin T \) with \( s_2 \in A, \quad s_1 \notin A. \)

If S is textured by T we call \( (S, T) \) a texture space or simply a texture. For a texture \( (S, T) \), most properties are conveniently defined in terms of the p-sets \( P_s = \cap \{ A \in T \mid s \in A \} \) and the q-sets, \( Q_s = \cup \{ A \in T / s \notin A \} \): The following are some basic examples of textures.

Examples 1.1. Some examples of texture spaces,

(1) If X is a set and \( P(X) \) the powerset of X, then \( (X; P(X)) \) is the discrete texture on X. For \( x \in X, \quad P_x = \{ x \} \) and \( Q_x = X \setminus \{ x \}. \)

(2) Setting \( I = [0; 1], \quad T = \{ \{0; r\}; \{0; r\} / r \in I \} \) gives the unit interval texture \( (I; T) \). For \( r \in I, \quad P_r = [0; r] \)

and \( Q_r = [0; r). \)

(3) The texture \( (L; T) \) is defined by \( L = (0; 1), \quad T = \{ \{0; r\} / r \in I \} \) \( \text{For } r \in L, \quad P_r = [0; r] \)

and \( Q_r = [0; r). \)

(4) \( T = \{ \phi, \{a, b\}, \{b\}, \{b, c\}, S \} \) is a simple texturing of \( S = \{a, b, c\} \) clearly \( P_a = \{a, b\}, \quad P_b = \{b\} \) and \( P_c = \{b, c\}. \)

Since a texturing T need not be closed under the operation of taking the set complement, the notion of topology is replaced by that of dichotomous topology or ditopology, namely a pair \( (\tau, \kappa) \) of subsets of T, where the set of open sets \( \tau \) satisfies

1. \( S, \phi \in \tau \)

2. \( G_1, G_2 \in \tau \) then \( G_1 \cap G_2 \in \tau \)

3. \( G_i \in \tau, \quad i \in I \) then \( \bigcup_{i \in I} G_i \in \tau, \)

and the set of closed sets \( \kappa \) satisfies

1. \( S, \phi \in \kappa \)

2. \( K_1, K_2 \in \kappa \) then \( K_1 \cup K_2 \in \kappa \) and

3. \( K_i \in \kappa, \quad i \in I \) then \( \bigcap_{i \in I} K_i \in \kappa. \) Hence a ditopology is essentially a "topology" for which there is no a priori relation between the open and closed sets.
For $A \in T$ we define the closure $[A]$ or $cl(A)$ and the interior $]A[$ or $int(A)$ under $(\tau, \kappa)$ by the equalities $[A] = \cap \{ K \in \kappa / A \subseteq K \}$ and $]A[ = \cup \{ G \in \tau / G \subseteq A \}$

**Definition 1.2.** For a ditopological texture space $(S; T; \tau, \kappa)$:

1. $A \in T$ is called pre-open (resp. semi-open, $\beta$-open) if $A \subseteq intcl A$ (resp. $A \subseteq intcl A$; $A \subseteq clintcl A$).
2. $B \in T$ is called pre-closed (resp. semi-closed, $\beta$-closed) if $clint B \subseteq B$ (resp. $intcl B \subseteq B$; $intclint B \subseteq B$).

We denote by $PO(S; T; \tau, \kappa)$ (respectively $\beta O(S; T; \tau, \kappa)$) the set of pre-open (respectively $\beta$-open) sets in $S$. Likewise, $PC(S; T; \tau, \kappa)$ (respectively $\beta C(S; T; \tau, \kappa)$) will denote the set of pre-closed (respectively $\beta$-closed) sets.

As in [3] we consider the sets $Q_s \in T$, $s \in S$, defined by $Q_s = V \{ P_t | s \notin P_t \}$. By [1.1] examples, we have $Q_x = X/\{ x \}$, $Q_r = (0, r]$ = $Pr$ and $Q_t = [0, t)$ respectively. The second example shows clearly that we can have $s \in Q_s$, and indeed even $Q_s \subseteq S$.

Also, in general, the sets $Q_s$ do not have any clear relation with either the set theoretic complement or the complementation on $T$. They are, however, closely connected with the notion of the core of the sets in $S$.

**Definition 1.3** For $A \in T$ the core of $A$ is the set $core(A) = \{ s \in S | A \nsubseteq Q_s \}$.

Clearly $core(A) \subseteq A$, and in general we can have $core(A) \nsubseteq T$. We will generally denote $core(A)$ by $Ab$.

2. **Dicovers and $\alpha$-locally finite**

**Definition 2.1** A subset $C$ of $T \times T$ is called a difamily on $(S, T)$. Let $C = \{ (G_\alpha, F_\alpha) | \alpha \in A \} \subseteq A$ be a family on $(S, T)$. Then $T$ is called a dicover of $(S, T)$ if for all partitions $A_1, A_2$ of $A$, we have $\cap \alpha \in A_1 F_\alpha \subseteq V \alpha \in A_2 G_\alpha$.

**Definition 2.2** Let $(\tau, \kappa)$ be a ditopology on $(S, T)$. Then $C$ is called $\alpha$-open ($\alpha$-open) if $dom(C) \subseteq \alpha O(S)$ (resp. $ran(C) \subseteq \alpha O(S)$).

**Definition 2.3** Let $(\tau, \kappa)$ be a ditopology on $(S, T)$. Then $C$ is called $\alpha$-closed ($\alpha$-closed) if $dom(C) \subseteq \alpha C(S)$ (resp. $ran(C) \subseteq \alpha C(S)$).

**Lemma 2.4** [3] Let $(S, T)$ be a texture. Then $P = \{ (P_s, Q_s) | s \in S \}$ is a dicover of $S$.

**Corollary 2.5** [3] Given $A \in T$, $A = \emptyset$, there exists $s \in S$ with $P_s \subseteq A$.

**Definition 2.6** Let $(S, T)$ be a texture, $C$ and $C'$ difamilies in $(S, T)$. Then $C$ is said to be a refinement of $C'$, written $C < C'$, if all $s \in S$ with $Q_s \neq S$ there exists $H_s \in H_s C$ (resp. $H_s C'$) so that $H_s \subseteq Q_s$. Also, it is the meet of $C$ and $C'$ with respect to the refinement relation.

**Definition 2.8** Let $C = \{ (G_i, F_i) | i \in I \}$ be a difamily indexed over $I$. Then $C$ is said to be (i) finite ($\alpha$-finite) if $dom(C)$ (resp. $ran(C)$) is finite.

(ii) $\alpha$-Locally finite if for all $s \in S$ there exists $K_s \in \alpha C(S)$ with $P_s \nsubseteq K_s$ so that the set $\{ i | G_i \nsubseteq K_s \}$ is finite.

(iii) $\alpha$-Locally co-finite if for all $s \in S$ with $Q_s \neq S$ there exists $H_s \in \alpha O(S)$ with $H_s \nsubseteq Q_s$ so that the set $\{ i | H_s \nsubseteq F_i \}$ is finite.

(iv) Point finite if for each $s \in S$ the set $\{ i | P_s \nsubseteq G_i \}$ is finite.

(v) Point co-finite if for each $s \in S$ with $Q_s \neq S$ the set $\{ i | F_i \nsubseteq Q_s \}$ is finite.

**Lemma 2.9** Let $(S, T, \tau, \kappa)$ be a ditopological texture space and $C$ be a difamily then, the following are equivalent:

1. $C = \{ (G_i, F_i) | i \in I \}$ is $\alpha$-locally finite.
2. There exists a family \( B = \{ B_j \mid j \in J \} \subset \mathcal{T}/\{\emptyset\} \) with the property that for \( A \in \mathcal{T} \) with \( A \neq \emptyset \), we have \( j \in J \) with \( B_j \subset A \), and for each \( j \in J \) there is \( K_j \in \alpha C(S) \) so that \( B_j \subset K_j \) and the set \( \{ i \mid Gi \subset K_j \} \) is finite.

Proof. Straightforward.

Lemma 2.10 Let \((S, T, \tau, \kappa)\) be a ditopological texture space and \( C \) be a difamily then, the following are equivalent:

(a) \( C = \{(Gi, Fi)\mid i \in I\} \) is \( \alpha \)-locally co-finite.

(b) There exists a family \( B = \{ B_j \mid j \in J \} \subset \mathcal{T}/\{\emptyset\} \) with the property that for \( A \in \mathcal{T} \) with \( A \neq \emptyset \), we have \( j \in J \) with \( A \subset B_j \), and for each \( j \in J \) there is \( H_j \in \alpha O(S) \) so that \( H_j \subset B_j \) and the set \( \{ i \mid Hi \subset F_j \} \) is finite.

Theorem 2.11 The difamily \( C = \{(Gi, Fi)\mid i \in I\} \) is \( \alpha \) locally finite if for each \( s \in S \) with \( Qs \neq S \) we have \( Ks \in \alpha C(S) \) with \( Ps \subset Ks \), so that the set \( \{ i \mid Gi \subset Ks \} \) is finite.

Proof. Given \( C = \{(Gi, Fi)\mid i \in I\} \) is \( \alpha \) locally finite, then by Lemma 2.9 there exists a family \( B = \{ B_j \mid j \in J \} \subset \mathcal{T}/\{\emptyset\} \) with the property that for \( A \in \mathcal{T} \) with \( A \neq \emptyset \), we have \( j \in J \) with \( A \subset B_j \), and for each \( j \in J \) there is \( K_j \in \alpha C(S) \) so that \( B_j \subset K_j \) and the set \( \{ i \mid Gi \subset K_j \} \) is finite. Now take \( B = \{ Ps \mid Qs \neq S \} = \{ Ps \mid s \in Sb \} \), and for \( A \in \mathcal{T} \) and \( A \neq \emptyset \), then by corollary 2.5 there exists \( s \in Sb \) with \( Ps \subset A \).

Therefore for every \( s \in S \), there exists \( Ks \in \alpha C(S) \) with \( Ps \subset Ks \) such that \( \{ i \mid Gi \subset Ks \} \) is finite.

Theorem 2.12 Let \((S, T, \tau, \kappa)\) be a ditopological texture space and \( C \) be a \( \alpha \)-locally finite dicover and \( s \in S \). Then there exists \( A \subset S \) and \( A \neq B \).

Proof. Given \( C = \{(Ai, Bi)\mid i \in I\} \) be a \( \alpha \)-locally finite. Take \( K \in \alpha C(S) \) with \( s \notin K \) and \( \{ i \mid Ai \subset K \} \) is finite. Now partition the set \( I \) into two sets such that \( I1 = \{ i \in I \mid s \notin Ai \} \) and \( I2 = \{ i \in I \mid s \notin Ai \), since \( C \) is a dicover it should satisfy \( \forall i \in I \) \( Bi \subset Vi \in I2 Ai \) now \( Vi \in I2 Ai \) does not have \( s \) according to our partition, which implies \( s \notin \forall i \in I1 Bi \). Thus we arrived at for all \( i \in I \) \( s \notin Ai \) and \( s \notin Bi \). (i.e) \( s \in A \) and \( s \notin B \).

Theorem 2.13 Let \((S, T, \tau, \kappa)\) be a ditopological texture space and \( C = \{(Ai, Bi)\mid i \in I\} \) be a difamily.

(1) If \( C \) is \( \alpha \)-locally finite, then \( \text{dom}(C) \) is \( \alpha \) closure preserving.

(2) If \( C \) is \( \alpha \)-locally co-finite, then \( \text{ran}(C) \) is \( \alpha \) interior preserving.

Proof. (1) Let \( I0 \) subset of \( I \). We have to prove \( \text{acl}(Vi \in I0 (Ai)) = Vi \in I0 \text{ acl}(Ai) \). To prove \( acl(Vi \in I0 (Ai)) \subset Vi \in I0 acl(Ai) \) suppose this is not true, we get there exists \( s \in S \) with \( s \in acl(Vi \in I0 (Ai)) \) and \( s \notin Vi \in I0 acl(Ai) \)

\[ acl(Vi \in I0 Ai) \subset Qs \quad \text{and} \quad Ps \subset Vi \in I0 acl(Ai) \]

Since \( C \) is \( \alpha \)-locally finite, we have \( \{ i \in I \mid Ai \subset Vi \in I0 acl(Ai) \} \) is finite. Now partition \( I0 \) into two sets such that

\[ I1 = \{ i \in I0 \mid Ai \subset K \} \quad \text{and} \quad I2 = I0/I1 \]

Now \( Vi \in I0 Ai = (Vi \in I2 \cup Vi \in I1)Ai = (Vi \in I2 \cup \forall i \in I1)Ai \)

\[ acl(Vi \in I0 Ai) \subset K \cup \forall i \in I1 Ai \]

using * we can say \( Vi \in I0 acl(Ai) \subset Qs \), which is a contradiction. Therefore \( acl(Vi \in I0 (Ai)) = Vi \in I0 acl(Ai) \). Hence \( \alpha \) closure is preserving.

(2) It is the dual of (1).

REFERENCE

On A Locally Finite In Ditopological...


