Integral Solutions of the Homogeneous Cubic Equation

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ABSTRACT:

The cubic equation \( x^3 + y^3 + xy(x + y) - z^3 - w^3 - zw(z + w) = (x + y + z + w)X^2 \) is analysed for its non-zero integral solutions. A few interesting relations between the solutions and special numbers are exhibited.

Keywords: Homogeneous equation with five unknowns, Integral solutions.

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NOTATIONS:

<table>
<thead>
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<th>Special Number</th>
<th>Notations</th>
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<tr>
<td>Gnomonic number</td>
<td>Gₙ</td>
<td>2n - 1</td>
</tr>
<tr>
<td>Pronic number</td>
<td>Pₙ</td>
<td>n(n + 1)</td>
</tr>
<tr>
<td>Star number</td>
<td>Sₙ</td>
<td>6n(n - 1) + 1</td>
</tr>
<tr>
<td>Octahedral number</td>
<td>OHₙ</td>
<td>( n(2n^2 + 1) )</td>
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I. INTRODUCTION

Integral solutions for the homogeneous or non-homogeneous Diophantine cubic equations is an interesting concept as it can be seen from [1-2]. In [3-13] a few special cases of cubic Diophantine equations with 4 unknowns are studied. In [14-15], the cubic equation with five unknowns is studied for its non-zero integral solutions. This communication concerns with another interesting cubic equation with five unknowns given by

\[
X^3 + Y^3 + XY(X + Y) - Z^3 - W^3 - ZW(Z + W) = (X + Y + Z + W)X^2
\]

for determining its integral solutions. A few interesting relations between the solutions are presented.

II. METHOD OF ANALYSIS

The cubic Diophantine equation with five unknowns to be solved for getting non-zero integral solutions is

\[
x^3 + y^3 + xy(x + y) - z^3 - w^3 - zw(z + w) = (x + y + z + w)X^2
\]  

(1)

On substituting the linear transformations

\[
x = u + v, y = u - v, z = u + p, w = u - p
\]  

(2)

in (1) leads to

\[
v^2 - p^2 = X^2
\]  

(3)

2.1 Pattern 1:

Equation (3) can be written as,

\[
v^2 = p^2 + X^2
\]  

(4)
which is satisfied by
\[
\begin{align*}
    v &= m^2 + n^2, \quad p = 2mn, \quad X = m^2 - n^2 \\
    v &= m^2 + n^2, \quad p = m^2 - n^2, \quad X = 2mn
\end{align*}
\]
\(m > n > 0\) \hspace{1cm} (5)

Substituting (5) in (2), the two sets of solutions satisfying (1) are obtained as follows:

\textbf{SET 1:}
\begin{align*}
    x(u, m, n) &= u + m^2 + n^2 \\
    y(u, m, n) &= u - m^2 - n^2 \\
    z(u, m, n) &= u + 2mn \\
    w(u, m, n) &= u - 2mn \\
    X(m, n) &= m^2 - n^2
\end{align*}

\textbf{SET 2:}
\begin{align*}
    x(u, m, n) &= u + m^2 + n^2 \\
    y(u, m, n) &= u - m^2 - n^2 \\
    z(u, m, n) &= u + m^2 - n^2 \\
    w(u, m, n) &= u - m^2 + n^2 \\
    X(m, n) &= 2mn
\end{align*}

2.2 Pattern 2:

Equation (3) can be written as
\[p^2 + X^2 = v^2 \ast 1\] \hspace{1cm} (6)

Assume \(v = a^2 + b^2\) \hspace{1cm} (7)

Write 1 as
\[1 = \frac{(1 + i)^{2n} (1 - i)^{2n}}{2^{2n}}\] \hspace{1cm} (8)

Substituting (7) and (8) in (6) and using the method of factorization, define
\[p + iX = (a + ib)^2 \frac{(1 + i)^{2n}}{2^n}\] \hspace{1cm} (9)

Equating real and imaginary parts of (9), we have
\[
\begin{align*}
    p &= (a^2 - b^2) \cos \frac{n\pi}{2} - 2ab \sin \frac{n\pi}{2} \\
    X &= (a^2 - b^2) \sin \frac{n\pi}{2} + 2ab \cos \frac{n\pi}{2}
\end{align*}
\]
Integral Solutions of the Homogeneous Cubic Equation

The corresponding integral values of \( X, y, z, w \) and \( X \) satisfying (1) are obtained as,

\[
\begin{align*}
  x(u, a, b) &= u + a^2 + b^2 \\
  y(u, a, b) &= u - a^2 - b^2 \\
  z(u, a, b) &= u + (a^2 - b^2) \cos \frac{n \pi}{2} - 2ab \sin \frac{n \pi}{2} \\
  w(u, a, b) &= u - (a^2 - b^2) \cos \frac{n \pi}{2} + 2ab \sin \frac{n \pi}{2} \\
  X(a, b) &= (a^2 - b^2) \sin \frac{n \pi}{2} + 2ab \cos \frac{n \pi}{2}
\end{align*}
\]

Properties:
1. Each of the following expression is a nasty number
   \[
   \begin{align*}
   i. \quad & 3 \left[ x(u, a, b) - y(u, a, b) \right]^2 + (-1)^{n-1} \left[ 12x(u, a, b)^2 - 3(z(u, a, b) - w(u, a, b))^2 \right] \\
   ii. \quad & 6 \left[ x(u, a, b) \times y(u, a, b) + (a^2 + b^2)^2 \right]
   \end{align*}
   \]
2. \( (2x(u, a, b), z(u, a, b) - w(u, a, b), x(u, a, b) - y(u, a, b)) \) forms a Pythagorean triple.
3. If \( a, b \) are taken as the generators of the Pythagorean triangle \( (\alpha, \beta, \gamma) \) whose sides are \( \alpha = a^2 - b^2; \beta = 2ab; \gamma = a^2 + b^2 \) then the product
\[
\left[ X(a, b) \sin \frac{n \pi}{2} + \left( \frac{z(u, a, b) - w(u, a, b)}{2} \right) \cos \frac{n \pi}{2} \right] \left[ X(a, b) \cos \frac{n \pi}{2} - \left( \frac{z(u, a, b) - w(u, a, b)}{2} \right) \sin \frac{n \pi}{2} \right]
\]
represents two times its area.
4. \( x(u, a, b)y(u, a, b) - z(u, a, b)w(u, a, b) \equiv 0(\text{mod} 2) \)
5. \( x(u, a, b) \pm y(u, a, b) \equiv 0(\text{mod} 2) \)

2.3 Pattern 3:

In (6) 1 can be written as
\[
1 = \frac{(p^2 + q^2 + i2pq)(p^2 - q^2 - i2pq)}{(p^2 + q^2)^2}; \quad \quad p > q > 0
\]

Proceeding as in Pattern II
\[
p + iX = (a + ib)^2 \frac{(p^2 + q^2 + i2pq)(p^2 - q^2 - i2pq)}{(p^2 + q^2)^2} \quad \quad (10)
\]

Equating real and imaginary parts,
\[
p = (a^2 - b^2) \frac{(p^2 - q^2)}{(p^2 + q^2)} - 4ab \frac{pq}{(p^2 + q^2)} \quad \quad (11)
\]
\[
X = (a^2 - b^2) \frac{2pq}{(p^2 + q^2)} + 2ab \frac{(p^2 - q^2)}{(p^2 + q^2)} \quad \quad (12)
\]
Since our aim is to find the integral solutions, substituting $a = (p^2 + q^2)A, b = (p^2 + q^2)B$ in (7), (11) and (12)

$$v = (p^2 + q^2)^2(A^2 + B^2)$$

$$p = (p^2 + q^2)[(A^2 - B^2)(p^2 - q^2) - 4ABpq]$$

$$X = 2pq(p^2 + q^2)(A^2 - B^2) + 2(p^2 + q^2)AB(p^2 - q^2)$$

Substituting (13), and (14) in (2) and using (15) we have the integral solutions of (1) as,

$$x(u, A, B) = u + (p^2 + q^2)^2(A^2 + B^2)$$

$$y(u, A, B) = u - (p^2 + q^2)^2(A^2 + B^2)$$

$$z(u, A, B) = u + (p^2 + q^2)(A^2 - B^2)(p^2 - q^2) - (p^2 + q^2)4ABpq$$

$$w(u, A, B) = u - (p^2 + q^2)(A^2 - B^2)(p^2 - q^2) + (p^2 + q^2)4ABpq$$

$$X(A, B) = 2pq(p^2 + q^2)(A^2 - B^2) + 2(p^2 + q^2)AB(p^2 - q^2)$$

Properties:

1. $4pq[z(u, p, q) - w(u, p, q)] - 2(p^2 + q^2)x(u, p, q) \equiv 0 \pmod{8}$

2. $2G_p(p^4 - q^4) + w(u, p, p - 1) - z(u, p, p - 1) \equiv 0 \pmod{8}$

3. $pq[z(u, p, p - 1) - w(u, p, p - 1)] - (p^2 - q^2)x(u, p, p - 1) + 2(p^2 + q^2)^3P_{p-1} = 0$

4. $2(p^2 + q^2)^2S_p + 3[y(u, p, p - 1) - x(u, p, p - 1)] \equiv 0 \pmod{4}$

5. $3OH_p + z(u, p, p) - w(u, p, p) - pq \equiv 0 \pmod{2}$

6. $w(u, p, q) - z(u, p, q) + x(u, p, q) + 10(p^2 + q^2)q^2$ is a perfect square.

7. Each of the following expression is a nasty number

i. $6u[x(u, A, B) + y(u, A, B) + z(u, A, B) + w(u, A, B)]$

ii. $6[x(u, A, A) - y(u, A, A)]$

III. CONCLUSION

To conclude one may search for other patterns of solutions and their corresponding properties.

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