Integral Solutions of the Non Homogeneous Ternary Quintic Equation

$$ax^2 - by^2 = (a-b)z^5, a,b > 0$$

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Abstract:

We obtain infinitely many non-zero integer triples (x, y, z) satisfying the ternary quintic equation $ax^2 - by^2 = (a-b)z^5, a,b > 0$. Various interesting relations between the solutions and special numbers, namely, polygonal numbers, Pyramidal numbers, Star numbers, Stella Octangular numbers, Pronic numbers, Octahedral numbers, Four Dimensional Figurative numbers and Five Dimensional Figurative numbers are exhibited.

Keywords: *Ternary quintic equation, integral solutions, 2-dimentional, 3-dimentional, 4- dimensional and 5- dimensional figurative numbers.*

MSC 2000 Mathematics subject classification: 11D41.

NOTATIONS:

$T_{m,n} = n\left(1 + \frac{(n-1)(m-2)}{2}\right)$ -Polygonal number of rank n with size m

$P_n^m = \frac{1}{6} n(n+1)((m-2)n + 5 - m)$ - Pyramidal number of rank n with size m

$SO_n = n(2n^2 - 1)$ -Stella octangular number ofrank n

$S_n = 6n(n-1) + 1$ -Star number of rank n

$PR_n = n(n+1)$ -Pronic number of rank n

$OH_n = \frac{1}{3}\left(n(2n^2 + 1)\right)$ - Octahedral number of rank n

$F_{5,n,3} = \frac{n(n+1)(n+2)(n+3)(n+4)}{5!}$ = Five Dimensional Figurative number of rank n whose generating polygon is a triangle.

$F_{4,n,3} = \frac{n(n+1)(n+2)(n+3)}{4!}$ = Four Dimensional Figurative number of rank n whose generating polygon is a triangle

1. INTRODUCTION

The theory of diophantine equations offers a rich variety of fascinating problems. In particular,quintic equations, homogeneous and non-homogeneous have aroused the interest of numerous mathematicians since antiquity[1-3]. For illustration, one may refer [4-10] for homogeneous and non-homogeneous quintic equations with three, four and five unknowns. This paper concerns with the problem of determining non-trivial integral solution of the non- homogeneous ternary quintic equation given by $ax^2 - by^2 = (a-b)z^5, a,b > 0$. A few relations between the solutions and the special numbers are presented.

II. METHOD OF ANALYSIS

The Diophantine equation representing the quintic equation with three unknowns under consideration is $ax^2 - by^2 = (a-b)z^5, a,b > 0$ (1)

Introduction of the transformations

$$x = X + bT, y = X + aT \quad (2)$$

in (1) leads to

$$X^2 - abT^2 = z^5 \quad (3)$$

The above equation (3) is solved through different approaches and thus, one obtains distinct sets of solutions to (1)

2.1. Case1:

Choose a and b such that ab is not a perfect square.

Assume $z = p^2 - abq^2$ (4)

Substituting (4) in (3) and using the method of factorisation, define

$$(X + \sqrt{ab}T) = (p + \sqrt{ab}q)^5 \quad (5)$$

Equating real and imaginary parts in (5) we get,

$$\left.\begin{aligned} X &= p^5 + 10abp^3q^2 + 5(ab)^2 pq^4 \\ T &= 5qp^4 + 10abp^2q^3 + (ab)^2 q^5 \end{aligned}\right\} \quad (6)$$

In view of (2) and (4), the corresponding values of x, y are represented by

$$x = p^5 + 10abp^3q^2 + 5(ab)^2 pq^4 + b(5qp^4 + 10abp^2q^3 + (ab)^2 q^5) \quad (7)$$

$$y = p^5 + 10abp^3q^2 + 5(ab)^2 pq^4 + a(5qp^4 + 10abp^2q^3 + (ab)^2 q^5) \quad (8)$$

Thus (7) and (4) represent the non-zero distinct integral solutions to (1)

A few interesting relations between the solutions of (1) are exhibited below:

1. $\left[2\frac{a(x(p,1) - by(p,1)}{a-b}\right] - 8T_{3,p-1} \times P_p^5 - (1+10ab)SO_p \equiv p(\text{mod}10)$

2. The following are nasty numbers;

(i) $60p\left\{2\frac{a(x(p,1) - by(p,1)}{a-b} - 8T_{3,p-1} \times P_p^5 - (1+10ab)SO_p + (9+10ab)p\right\}$

$(ii) 30p\left[\frac{ax(p,1) - by(p,1)}{a-b} - 120F_{5,p,3} + 240F_{4,p,3} - (2ab+5)30P_p^3 + 15PR_p + 12T_{4,p} - 8T_{5,p} + 60abT_{3,p}\right]$

3. $2ab\left\{5(T_{4,p} + ab)^2 - \left[\frac{x(p,1) - y(p,1)}{b-a}\right]\right\}$ is a cubical integer.

4. $ax(p,1) - by(p,1) + (a-b)\{3840F_{4,p,3} - 1920F_{5,p,3} - 1200P_p^4 - 720T_{3,p} - 48T_{4,p} + 32T_{5,p}\} \equiv 0(\text{mod}5)$

5. $\left[2\frac{a(x(p,1) - by(p,1)}{a-b}\right] - 8T_{3,p-1} \times P_p^5 - 3(1+10ab)OH_p + 11(2T_{3,p} - T_{4,P}) \equiv 0(\text{mod}10)$

6. $S_p - 6z(p,1) + 18(OH_p) - 6SO_p - 12T_{3,p} + 6T_{4,p} - 6ab = 1$

Now, rewrite (3) as,

$$X^2 - abT^2 = 1 \times z^5 \quad (9)$$

Also 1 can be written as

$$1 = \frac{(a+b+2\sqrt{ab})(a+b-2\sqrt{ab})}{(a-b)^2} \quad (10)$$

Substituting (4) and (10) in (9) and using the method of factorisation, define,

$$(X+\sqrt{ab}T)=\frac{(a+b+2\sqrt{ab})}{a-b}(p+\sqrt{ab}q)^5 \tag{11}$$

Equating real and imaginary parts in (11) we obtain,

$$X=\frac{1}{a-b}\left[(a+b)(p^5+10abp^3q^2+5(ab)^2pq^4)+2ab(5qp^4+10abq^3p^2+(ab)^2q^5)\right]$$

$$T=\frac{1}{a-b}\left[(a+b)(5qp^4+10abq^3p^2+(ab)^2q^5)+2(p^5+10abp^3q^2+5(ab)^2pq^4)\right]$$

In view of (2) and (4), the corresponding values of x, y and z are obtained as,

$$\left.\begin{aligned} x&=(a-b)^4[(a+3b)(p^5+10abp^3q^2+5(ab)^2pq^4)+\\ &\qquad b(3a+b)(5qp^4+10abp^2q^3+(ab)^2q^5)]\\ y&=(a-b)^4[(3a+b)(p^5+10abp^3q^2+5(ab)^2pq^4)+\\ &\qquad a(a+3b)(5qp^4+10abp^2q^3+(ab)^2q^2)]\\ z&=(a-b)^2(p^2-abq^2)\end{aligned}\right\} \tag{12}$$

Further 1 can also be taken as

$$1=\frac{(ab+\alpha^2+2\alpha\sqrt{ab})(ab+\alpha^2-2\alpha\sqrt{ab})}{(ab-\alpha^2)^2} \tag{13}$$

For this choice, after performing some algebra the value of x, y and z are given by

$$\left.\begin{aligned} x&=(ab-\alpha^2)^4[(ab+\alpha^2-2\alpha b)(p^5-10abp^3q^2+5(ab)^2pq^4)+\\ &\qquad (2\alpha ab-b(ab+\alpha^2))(5qp^4-10abp^2q^3+(ab)^2q^5)]\\ y&=(ab-\alpha^2)^4[(ab+\alpha^2-2\alpha a)(p^5-10abp^3q^2+5(ab)^2pq^4)+\\ &\qquad (2\alpha ab-a(ab+\alpha^2))(5qp^4-10abp^2q^3+(ab)^2q^5)]\\ z&=(ab-\alpha^2)^2(p^2-abq^2)\end{aligned}\right\} \tag{14}$$

Remark1:

Instead of (2), the introduction of the transformations, $x=X-bT, y=X-aT$ in (1) leads to (3).By a similar procedure we obtain different integral solutions to (1).

2.2. Case2:

Choose a and b such that ab is a perfect square,say, d^2.

$\therefore$ (3) is written as $X^2-(dT)^2=z^5$ (15)

To solve (15), we write it as a system of double equations which are solved in integers for X, T and in view of (2), the corresponding integral values of x, y are obtained. The above process is illustrated in the following table:

System of double equations	Integral values of z	Integral values of x, y
$X+dT=z^4$ $X-dT=z$	$z=2dk$	$x=8k^4d^3(d+b)+k(d-b)$ $y=8k^4d^3(d+a)+k(d-a)$
$X+dT=z$ $X-dT=z^4$	$z=2dk$	$x=8k^4d^3(d-b)+k(d+b)$ $y=8k^4d^3(d-a)+k(d+a)$

$X+dT=z^3$ $X-dT=z^2$	$z=2dk$	$x=4k^3d^2(d+b)+2dk^2(d-b)$ $y=4k^3d^2(d+a)+2dk^2(d-a)$
$X+dT=z^2$ $X-dT=z^3$	$z=2dk$	$x=4k^3d^2(d-b)+2dk^2(d+b)$ $y=4k^3d^2(d-a)+2dk^2(d+a)$
$X+dT=z^5$ $X-dT=1$	$z=2dk+1$	$x=(16k^5d^4+40k^4d^3+40k^3d^2+20k^2d+5k)(d+b)+1$ $y=(16k^5d^4+40k^4d^3+40k^3d^2+20k^2d+5k)(d+a)+1$
$X+dT=z^5$ $X-dT=1$	$z=2dk+1$	$x=(16k^5d^4+40k^4d^3+40k^3d^2+20k^2d+5k)(d-b)+1$ $y=(16k^5d^4+40k^4d^3+40k^3d^2+20k^2d+5k)(d-a)+1$

2.3. Case3:

Take $z=\alpha^2$ (16)

Substituting (16) in (15) we get,

$$X^2=(dT)^2+(\alpha^5)^2 \quad (17)$$

which is in the form of Pythagorean equation and is satisfied by

$$\alpha^5=2rs, dT=r^2-s^2, X=r^2+s^2, r>s>0 \quad (18)$$

Choose r and s such that $rs=16d^5k^5$ (19)

and thus $\alpha=2dk$.

Knowing the values of r,s and using (18) and (2), the corresponding solutions of (1) are obtained. For illustration, take

$$r=2^3d^4k^4, s=2dk, r>s>0$$

The corresponding solutions are given by

$$\left.\begin{aligned} x&=64d^7k^8(d+b)+4dk^2(d-b)\\ y&=64d^7k^8(d+a)+4dk^2(d-a)\\ z&=4d^2k^2 \end{aligned}\right\} \quad (20)$$

Taking the values of r,s differently such that $rs=16d^5k^5$, we get different solution patterns.

It is to be noted that, the solutions of (17) may also be written as

$$X=r^2+s^2, dT=2rs, \alpha^5=r^2-s^2, r>s>0 \quad (21)$$

Choose $r=\dfrac{\alpha^3+\alpha^2}{2}, s=\dfrac{\alpha^3-\alpha^2}{2}$

Substituting the values of r,s in (21) and performing some algebra using (2) we get the corresponding solutions as

$$\left.\begin{aligned} x&=32d^5k^6(d+b)+8d^3k^4(d-b)\\ y&=32d^5k^6(d+a)+8d^3k^4(d-a)\\ z&=4d^2k^2 \end{aligned}\right\} \quad (22)$$

Taking the values of r,s differently such that $\alpha^5=r^2-s^2$, we get different solution patterns.

2.4. Case4:

Also, introducing the linear transformations

$$X=2dk+u, dT=2dk-u$$

in (15) and performing a few algebra, we have

$$X = 4k^4d^4 + 2dk, T = 2k - 4k^4d^3 \quad (23)$$

In view of (2) and (23) the corresponding integral solutions of (1) are obtained as

$$\left.\begin{array}{l} x = 4k^4d^3(d-b) + 2k(d+b) \\ y = 4k^4d^3(d-a) + 2k(d+a) \\ z = 2dk \end{array}\right\} \quad (24)$$

It is to be noted that, in addition to the above choices for X and dT, we have other choices, which are illustrated below:

Illustration1:

The assumption $X = zX, dT = zdT$ (25)

in (15) yields

$$X = md^3(m^2 - n^2), T = nd^2(m^2 - n^2), z = d^2(m^2 - n^2) \quad (26)$$

From (25), (26) and (2), the corresponding integral solutions of (1) are

$$\left.\begin{array}{l} x = d^4(m^2 - n^2)^2(md + nb) \\ y = d^4(m^2 - n^2)^2(md + na) \\ z = d^2(m^2 - n^2) \end{array}\right\} \quad (26)$$

Illustration2:

The assumption $X = z^2X, dT = z^2dT$ (27)

in (15) yields to $X^2 - (dT)^2 = z$ (28)

Taking $z = 4m^2n^2d^2$ (29)

and performing some algebra, we get

$$X = 16m^4n^4d^{10}(m^2 + n^2), T = 16m^4n^4d^9(n^2 - m^2) \quad (30)$$

From (29), (30) and (2), the corresponding integral solutions of (1) are

$$\left.\begin{array}{l} x = 16m^4n^4d^9[d(m^2 + n^2) + b(n^2 - m^2)] \\ y = 16m^4n^4d^9[d(m^2 + n^2) + a(n^2 - m^2)] \\ z = 4m^2n^2d^2 \end{array}\right\} \quad (31)$$

Illustration3:

Instead of (29), taking $z = 2dk + 1$ (32)

in (28) we get

$$X = (2dk+1)^2(dk+1), T = -k(2k+1)^2 \quad (33)$$

From (30), (31) and (2), the corresponding integral solutions of (1) are

$$\left.\begin{array}{l} x = (2dk+1)^2(dk - bk + 1) \\ y = (2dk+1)^2(dk - ak + 1) \\ z = 2dk + 1 \end{array}\right\} \quad (34)$$

Illustration4:

In (28), taking $z = -t^2$ and arranging we get (35)

$$X^2 + t^2 = (dT)^2 \quad (36)$$

which is in the form of Pythagorean equation and is satisfied by

$$X = 2pq, dT = p^2 + q^2, t = p^2 - q^2, r > s > 0 \quad (37)$$

From (35), (37) and (2) we get the corresponding solutions as

$$\left.\begin{array}{l} x = (p^2 - q^2)^4d^9[2pqd + b(p^2 + q^2)] \\ y = (p^2 - q^2)^4d^9[2pqd + a(p^2 + q^2)] \\ z = -(p^2 - q^2)^2d^4 \end{array}\right\} \quad (38)$$

It is to be noted that, the solutions of (36) may also be written as

$$X = p^2 - q^2, t = 2pq, dT = p^2 + q^2, p > q > 0 \quad (39)$$

From (33),(37) and (2) we get the corresponding solutions as

$$\left.\begin{aligned} x &= 16p^4q^4d^9[d(p^2 - q^2) + b(p^2 + q^2)] \\ y &= 16p^4q^4d^9[d(p^2 - q^2) + a(p^2 + q^2)] \\ z &= -4p^2q^2d^4 \end{aligned}\right\} \quad (40)$$

III. Remarkable observations:

I: If (x_0, y_0, z_0) be any given integral solution of (1), then the general solution pattern is presented in the matrix form as follows:

Odd ordered solutions:

$$\begin{pmatrix} x_{2m-1} \\ y_{2m-1} \\ z_{2m-1} \end{pmatrix} = (a-b)^{4m-2} \begin{pmatrix} -(a+b)(a-b)^{6m-4} & +2a(a-b)^{6m-4} & 0 \\ -2a(a-b)^{6m-4} & (a+b)(a-b)^{6m-4} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$$

Even ordered solutions:

$$\begin{pmatrix} x_{2m} \\ y_{2m} \\ z_{2m} \end{pmatrix} = (a-b)^{4m} \begin{pmatrix} -(a+b)(a-b)^{6m} & +2b(a-b)^{6m} & 0 \\ -2a(a-b)^{6m} & (a+b)(a-b)^{6m} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$$

II: The integral solutions of (1) can also be represented as,

$$x = (m + bN)Z^2$$

$$y = (m + aN)Z^2$$

$$z = m^2 + abN^2, a, b > 0$$

IV. CONCLUSION

In conclusion, one may search for different patterns of solutions to (1) and their **c**orresponding properties.

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