On The Zeros of Polynomials and Analytic Functions

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Abstract
In this paper we obtain some results on the zeros of polynomials and related analytic functions, which generalize and improve upon the earlier well-known results.

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I. INTRODUCTION AND STATEMENT OF RESULTS

Regarding the zeros of a polynomial , Jain [2] proved the following results:

Theorem A: Let $P(z)=a_0+a_1z+......+a_{p-1}z^{p-1}+a_pz^p+......+a_nz^n$ be a polynomial of degree n such that $a_{p-1}\neq a_p$ for some $p\in\{1,2,......,\ n\}$,

$$M=M_p=\sum_{j=p+1}^{n}\left|a_j-a_{j-1}\right|+\left|a_n\right|,\qquad (1\le p\le n),\ M_n=\left|a_n\right|$$

$$m=m_p=\sum_{j=1}^{p-1}\left|a_j-a_{j-1}\right|,\qquad (2\le p\le n),\ m_1=0.$$

Then P(z) has at least p zeros in

$$\left|z\right|<\frac{p}{M}\frac{(a_p-a_{p-1})}{p+1},$$

provided

$$\frac{p}{M}\frac{(a_p-a_{p-1})}{p+1}<1$$

and

$$\left|a_0\right|+m\frac{p}{M}\left(\frac{a_p-a_{p-1}}{p+1}\right)<\left(\frac{p}{M}\right)^p\left(\frac{a_p-a_{p-1}}{p+1}\right).$$

Theorem B: Let $P(z)=a_0+a_1z+......+a_{p-1}z^{p-1}+a_pz^p+......+a_nz^n$ be a polynomial of degree n such that $a_{p-1}\neq a_p$ for some $p\in\{1,2,......,\ n\}$,

$$\left|\arg a_j-\beta\right|\le\alpha\le\frac{\pi}{2},\ j=0,1,.......,\ n,$$

for some real β and α and

$$\left|a_n\right|\ge\left|a_{n-1}\right|\ge......\ge\left|a_1\right|\ge\left|a_0\right|.$$

Then P(z) has at least p zeros in

$$\left|z\right|<\frac{p}{L}\frac{\left|a_p-a_{p-1}\right|}{p+1},$$

where

$$L=L_p=\left|a_n\right|+(\left|a_n\right|-\left|a_p\right|)\cos\alpha+\sum_{j=p+1}^{n}(\left|a_j\right|+\left|a_{j-1}\right|)\sin\alpha$$

and

$$l = l_p = (|a_{p-1}| - |a_0|)\cos\alpha + \sum_{j=1}^{p-1}(|a_j| + |a_{j-1}|)\sin\alpha \qquad (2 \le p \le n-1),\ l_1 = 0\ ,$$

provided

$$|a_0| + l\,\frac{p}{L}\frac{|a_p - a_{p-1}|}{p+1} < \left(\frac{p}{L}\right)^p \left(\frac{|a_p - a_{p-1}|}{p+1}\right)^{p+1}.$$

In this paper, we prove the following results:

Theorem 1: Let $P(z) = a_0 + a_1 z + ... + a_{p-1}z^{p-1} + a_p z^p + ... + a_n z^n$ be a polynomial of degree n such that $a_{p-1} \ne a_p$ for some $p \in \{1,2,......,\ n-1\}$ and $\rho \ge 0$,

$$\rho + a_n \ge a_{n-1} \ge ...... \ge a_p > a_{p-1} \ge ...... \ge a_1 \ge a_0 .$$

Then P(z) has at least p zeros in

$$\frac{|a_0|}{K(2\rho + a_n - a_0 + K^n|a_n|)} \le |z| \le K = \frac{p}{M}\frac{(a_p - a_{p-1})}{p+1},$$

where

$$M = 2\rho + |a_n| + a_n - a_p ,$$

provided

$$|a_0| + \frac{p}{M}\left(\frac{a_p - a_{p-1}}{p+1}\right)(a_{p-1} - a_0) < \left(\frac{p}{M}\right)^p \left(\frac{a_p - a_{p-1}}{p+1}\right)^{p+1}$$

and K<1.

Remark 1: Taking $\rho = 0$ in Theorem 1, we get the following result:

Corollary 1: Let $P(z) = a_0 + a_1 z + ... + a_{p-1}z^{p-1} + a_p z^p + ... + a_n z^n$ be a polynomial of degree n such that $a_{p-1} \ne a_p$ for some $p \in \{1,2,......,\ n-1\}$

$$a_n \ge a_{n-1} \ge ...... \ge a_p > a_{p-1} \ge ...... \ge a_1 \ge a_0 .$$

Then P(z) has at least p zeros in

$$\frac{|a_0|}{2\rho + a_n - a_0 + K_1^{\,n}|a_n|} \le |z| \le K_1 = \frac{p}{M_1}\frac{(a_p - a_{p-1})}{p+1},$$

where

$$M_1 = |a_n| + a_n - a_p ,$$

provided

$$|a_0| + \frac{p}{M_1}\left(\frac{a_p - a_{p-1}}{p+1}\right)(a_{p-1} - a_0) < \left(\frac{p}{M_1}\right)^p \left(\frac{a_p - a_{p-1}}{p+1}\right)^{p+1}$$

and $K_1 < 1$.

This result was earlier proved by Roshan Lal et al [4] .

If the coefficients are positive in Theorem 1, we have the following result:

Corollary 2: Let $P(z) = a_0 + a_1 z + ... + a_{p-1}z^{p-1} + a_p z^p + ... + a_n z^n$ be a polynomial of degree n such that $a_{p-1} \ne a_p$ for some $p \in \{1,2,......,\ n-1\}$ and $\rho \ge 0$,

$$\rho + a_n \ge a_{n-1} \ge ...... \ge a_p > a_{p-1} \ge ...... \ge a_1 \ge a_0 > 0 .$$

Then P(z) has at least p zeros in

$$\frac{a_0}{K_2\{2\rho - a_0 + (K_2^{\,n} + 1)a_n)\}} \le |z| \le K_2 = \frac{p}{M_2}\frac{(a_p - a_{p-1})}{p+1},$$

where

$$M_2 = 2(\rho + a_n) - a_p ,$$

provided

$$|a_0| + \frac{p}{M_2}\left(\frac{a_p - a_{p-1}}{p+1}\right)(a_{p-1} - a_0) < \left(\frac{p}{M_2}\right)^p \left(\frac{a_p - a_{p-1}}{p+1}\right)^{p+1}$$

and $K_2 < 1$.

If the coefficients of the polynomial P(z) are complex, we prove the following result:

Theorem 2: Let $P(z) = a_0 + a_1 z + ... + a_{p-1} z^{p-1} + a_p z^p + ... + a_n z^n$ be a polynomial of degree n such that $a_{p-1} \neq a_p$ for some $p \in \{1,2,......, \quad n-1\}$ and $\rho \geq 0$,

$$|\rho + a_n| \geq |a_{n-1}| \geq ...... \geq |a_p| > |a_{p-1}| \geq ...... \geq |a_1| \geq |a_0|$$

and for some real β and α ,

$$|\arg a_j - \beta| \leq \alpha \leq \frac{\pi}{2}, \; j = 0,1,......, n.$$

Then P(z) has at least p zeros in

$$|z| < K_3 = \frac{p}{M_3}\left(\frac{|a_p - a_{p-1}|}{p+1}\right),$$

where

$$M_3 = (\rho + |a_n|)(\cos \alpha + \sin \alpha + 1) - |a_p| \cos \alpha + \sum_{j=p+1}^{n-1} (|a_j| + |a_{j-1}|) \sin \alpha ,$$

provided

$$|a_0| + \frac{p}{M_3}\left(\frac{|a_p - a_{p-1}|}{p+1}\right) m' < \left(\frac{p}{M_3}\right)^p \left(\frac{|a_p - a_{p-1}|}{p+1}\right)^{p+1}$$

Remark 2: Taking $\rho = 0$ in Theorem 2, it reduces to Theorem B.

Remark 3: It is easy to see that K_3 <1.

Next, we prove the following result on the zeros of analytic functions :

Theorem 3: Let $f(z) = \sum_{j=0}^{\infty} a_j z^j \neq 0$ be analytic in $|z| \leq K_4$ and for some natural number p with

$$\frac{a_{p-1}}{a_p} < 2 + \frac{1}{p},$$

$$\rho + a_0 \geq a_1 \geq a_2 \geq ...... \geq a_{p-1} > a_p \geq a_{p+1} \geq ...... ,$$

for some $\rho \geq 0$,and

$$|a_0| + \frac{p}{a_p}\left(\frac{a_{p-1} - a_p}{p+1}\right)(2\rho + a_0 - a_{p-1}) < \left(\frac{p}{a_p}\right)^p \left(\frac{a_p - a_{p-1}}{p+1}\right)^{p+1}.$$

Then f(z) has at least p zeros in

$$|z| < K_4 = \frac{p}{p+1}\left(\frac{a_{p-1} - a_p}{a_p}\right).$$

Remark 4: Taking $\rho = 0$, Theorem 3 reduces to the following result:

Corollary 3: Let $f(z) = \sum_{j=0}^{\infty} a_j z^j \neq 0$ be analytic in $|z| \leq K_4$ and for some natural number p with

$$\frac{a_{p-1}}{a_p} < 2 + \frac{1}{p},$$

$$\rho + a_0 \geq a_1 \geq a_2 \geq ...... \geq a_{p-1} > a_p \geq a_{p+1} \geq ......,$$

for some $\rho \geq 0$,and

$$|a_0| + \frac{p}{a_p}\left(\frac{a_{p-1} - a_p}{p+1}\right)(a_0 - a_{p-1}) < \left(\frac{p}{a_p}\right)^p \left(\frac{a_p - a_{p-1}}{p+1}\right)^{p+1}.$$

Then f(z) has at least p zeros in

$$|z| < K_4 = \frac{p}{p+1}\left(\frac{a_{p-1} - a_p}{a_p}\right).$$

Cor.3 was earlier proved by Roshan Lal et al [4].

II. LEMMA

For the proofs of the above results , we need the following lemma due to Govil and Rahman [1]:

Lemma : If a_1 and a_2 are complex numbers such that

$$|\arg a_j - \beta| \leq \alpha \leq \frac{\pi}{2},\ j = 1,2, \text{ for some real numbers } \beta \text{ and } \alpha,$$

then

$$|a_1 - a_2| \leq (|a_1| - |a_2|)\cos\alpha + (|a_1| + |a_2|)\sin\alpha .$$

III. PROOFS OF THE THEOREMS

3.1 Proof of Theorem 1: Consider the polynomial

$$\begin{aligned}
F(z) &= (1-z)P(z) \\
&= (1-z)(a_0 + a_1 z + ... + a_{p-1}z^{p-1} + a_p z^p + ... + a_n z^n) \\
&= a_0 + \sum_{j=1}^{p-1}(a_j - a_{j-1})z^j + (a_p - a_{p-1})z^p + \sum_{j=p+1}^{n}(a_j - a_{j-1})z^j \\
&\quad - a_n z^{n+1} \\
&= \phi(z) + \psi(z),
\end{aligned}$$

where

$$\phi(z) = a_0 + \sum_{j=1}^{p-1}(a_j - a_{j-1})z^j$$

and

$$\psi(z) = (a_p - a_{p-1})z^p + \sum_{j=p+1}^{n}(a_j - a_{j-1}) - a_n z^{n+1}.$$

For $|z| = K$ (<1), we have, by using the hypothesis,

$$\begin{aligned}
|\psi(z)| &\geq |a_p - a_{p-1}|K^p - \left(\sum_{j=p+1}^{n}|a_j - a_{j-1}|K^j + |a_n|K^{n+1}\right) \\
&\geq (a_p - a_{p-1})K^p - K^{p+1}\left(|a_n|K^{n-p} + |a_n - a_{n-1}|K^{n-p-1} + \sum_{j=p+1}^{n-1}|a_j - a_{j-1}|K^{n-(p+1)}\right) \\
&\geq (a_p - a_{p-1})K^p - K^{p+1}\left(|a_n| + |\rho + a_n - a_{n-1} - \rho| + a_{n-1} - a_p\right) \\
&\geq (a_p - a_{p-1})K^p - K^{p+1}\left(|a_n| + \upsilon + a_n - a_{n-1} + \rho + a_{n-1} - a_p\right) \\
&\geq (a_p - a_{p-1})K^p - K^{p+1}\left(2\rho + |a_n| + a_n - a_p\right)
\end{aligned}$$

$$= (a_p - a_{p-1})\left[\frac{p(a_p - a_{p-1})}{2\rho + |a_n| + a_n - a_p}\right]^p$$

$$- \left[\frac{p(a_p - a_{p-1})}{2\rho + |a_n| + a_n - a_p}\right]^{p+1} \left(2\rho + |a_n| + a_n - a_p\right)$$

$$= \left[\frac{p}{2\rho + |a_n| + a_n - a_p}\right]^p \left[\frac{a_p - a_{p-1}}{p+1}\right]^{p+1}$$

$$= \left[\frac{p}{M}\right]^p \left[\frac{a_p - a_{p-1}}{p+1}\right]^{p+1} \qquad (1)$$

Also for $|z| = K$ (<1),

$$|\phi(z)| \le |a_0| + \sum_{j=1}^{p-1} |a_j - a_{j-1}| K^j$$

$$\le |a_0| + K \sum_{j=1}^{p-1} (a_j - a_{j-1})$$

$$\le |a_0| + K(a_{p-1} - a_0)$$

$$= |a_0| + \left(\frac{p(a_p - a_{p-1})}{2\rho + |a_n| + a_n - a_p}\right)\left(\frac{a_{p-1} - a_0}{p+1}\right)$$

$$= |a_0| + \frac{p}{M}\left(\frac{a_p - a_{p-1}}{p+1}\right)\left(a_{p-1} - a_0\right) \qquad (2)$$

Since, by hypothesis

$$|a_0| + \frac{p}{M}\left(\frac{a_p - a_{p-1}}{p+1}\right)\left(a_{p-1} - a_0\right) < \left(\frac{p}{M}\right)^p \left(\frac{a_p - a_{p-1}}{p+1}\right)^{p+1},$$

it follows from (1) and (2) that

$|\psi(z)| < |\phi(z)|$ for $|z| = K$.

Hence, by Rouche's theorem, $\phi(z)$ and $\phi(z) + \psi(z)$ i.e. F(z) have the same number of zeros in $|z| < K$

Since the zeros of P(z) are also the zeros of F(z) and since $\psi(z)$ has at least p zeros in $|z| < K$,it follows that P(z) has at least p zeros in $|z| < K$ .That prove the first part of Theorem 1.

To prove the second part, we show that P(z) has no zero in

$$|z| < \frac{|a_0|}{2\rho + a_n - a_0 + |a_n| K^n}.$$

Let $F(z) = (1 - z)P(z)$

$$= (1 - z)(a_0 + a_1 z + \ldots\ldots + a_{p-1} z^{p-1} + a_p z^p + \ldots\ldots + a_n z^n)$$

$$= a_0 + \sum_{j=1}^{n} (a_j - a_{j-1}) z^j - a_n z^{n+1}$$

$$= a_0 + g(z),$$

where

$$g(z) = \sum_{j=1}^{n1} (a_j - a_{j-1}) z^j - a_n z^{n+1}.$$

For $|z| = K$ (<1), we have, by using the hypothesis,

$$|g(z)| \le \sum_{j=1}^{n} |a_j - a_{j-1}| |z|^j + |a_n| |z|^{n+1}$$

$$= \sum_{j=1}^{n} |a_j - a_{j-1}| K^j + |a_n| K^{n+1}$$

$$\le K \left[|a_n - a_{n-1}| + \sum_{j=1}^{n-1} (a_j - a_{j-1}) + |a_n| K^n \right]$$

$$= K \left[|\rho + a_n - a_{n-1} - \rho| + a_{n-1} - a_0 + |a_n| K^n \right]$$

$$\le K \left[\rho + a_n - a_{n-1} + \rho + a_{n-1} - a_0 + |a_n| K^n \right]$$

$$\le K \left[2\rho + a_n - a_0 + |a_n| K^n \right].$$

Since g(z) is analytic for $|z| \le K$, g(0)=0, we have, by Schwarz's lemma,

$$|g(z)| \le K \left[2\rho + a_n - a_0 + |a_n| K^n \right] |z| \quad \text{for } |z| \le K.$$

Hence, for $|z| \le K$,

$$|F(z)| = |a_0 + g(z)|$$

$$\ge |a_0| - |g(z)|$$

$$\ge |a_0| - K \left[2\rho + a_n - a_0 + |a_n| K^n \right] |z|$$

$$> 0$$

if

$$|z| < \frac{|a_0|}{K(2\rho + a_n - a_0 + |a_n| K^n)}.$$

This shows that F(z) and therefore P(z) has no zero in

$$|z| < \frac{|a_0|}{K(2\rho + a_n - a_0 + |a_n| K^n)}.$$

That proves Theorem 1 completely.

Proof of Theorem 2: Consider the polynomial

$$F(z) = (1 - z) P(z)$$

$$= (1 - z)(a_0 + a_1 z + \ldots\ldots + a_{p-1} z^{p-1} + a_p z^p + \ldots\ldots + a_n z^n)$$

$$= a_0 + \sum_{j=1}^{p-1} (a_j - a_{j-1}) z^j + (a_p - a_{p-1}) z^p + \sum_{j=p+1}^{n} (a_j - a_{j-1}) z^j$$

$$- a_n z^{n+1}$$

$$= \phi(z) + \psi(z),$$

where

$$\phi(z) = a_0 + \sum_{j=1}^{p-1} (a_j - a_{j-1}) z^j$$

and

$$\psi(z)=(a_p-a_{p-1})z^p+\sum_{j=p+1}^{n}(a_j-a_{j-1})-a_nz^{n+1}..$$

For $|z|=K_3$ (<1), we have, by using the hypothesis,

$$|\psi(z)|\geq|a_p-a_{p-1}|K_3^{\,p}-\left(\sum_{j=p+1}^{n}|a_j-a_{j-1}|K_3^{\,j}+|a_n|K_3^{\,n+1}\right)$$

$$\geq|a_p-a_{p-1}|K_3^{\,p}-K_3^{\,p+1}\left(|a_n|K_3^{\,n-p}+|a_n-a_{n-1}|K_3^{\,n-p-1}+\sum_{j=p+1}^{n-1}|a_j-a_{j-1}|K_3^{\,n-(p+1)}\right)$$

$$>|a_p-a_{p-1}|K_3^{\,p}-K_3^{\,p+1}\left(|a_n|+|\rho+a_n-a_{n-1}-\rho|+\sum_{j=p+1}^{n-1}|a_j-a_{j-1}|\right)$$

$$\geq|a_p-a_{p-1}|K_3^{\,p}-K_3^{\,p+1}\left(|a_n|+|\rho+a_n-a_{n-1}|+\rho+\sum_{j=p+1}^{n-1}|a_j-a_{j-1}|\right)$$

$$\geq|a_p-a_{p-1}|K_3^{\,p}-K_3^{\,p+1}\big[\rho+|a_n|+\{(\rho+|a_n|)-|a_{n-1}|\}\cos\alpha+$$

$$\{(\rho+|a_n|)+|a_{n-1}|\}\sin\alpha+(|a_{p+1}|-|a_p|)\cos\alpha+(|a_{p+1}|+|a_p|)\sin\alpha$$

$$+\ldots\ldots+(|a_{n-1}|-|a_{n-2}|)\cos\alpha+(|a_{n-1}|+|a_{n-2}|)\sin\alpha\ \big]$$

$$=|a_p-a_{p-1}|K_3^{\,p}-K_3^{\,p+1}\big[(\rho+|a_n|)(\cos\alpha+\sin\alpha+1)-|a_p|\cos\alpha$$

$$+\sum_{j=p+1}^{n-1}(|a_j|+|a_{j-1}|)\sin\alpha\ \big]$$

$$=|a_p-a_{p-1}|K_3^{\,p}-K_3^{\,p+1}M_3$$

$$=|a_p-a_{p-1}|\left(\frac{p}{M_3}\frac{|a_p-a_{p-1}|}{p+1}\right)^p-\left(\frac{p}{M_3}\frac{|a_p-a_{p-1}|}{p+1}\right)^{p+1}M_3$$

$$=\left(\frac{p}{M_3}\right)^p\left(\frac{|a_p-a_{p-1}|}{p+1}\right)^{p+1}$$

$$>|a_0|+\frac{p}{M_3}\left(\frac{|a_p-a_{p-1}|}{p+1}\right)m'$$

$$=|a_0|+K_3m' \qquad (3)$$

Also , for $|z|=K_3$, we have, by using the lemma and the hypothesis,

$$|\phi(z)|\leq|a_0|+\sum_{j=1}^{p-1}|a_j-a_{j-1}||z|^j$$

$$=|a_0|+\sum_{j=1}^{p-1}|a_j-a_{j-1}|K_3^{\,j}$$

$$<|a_0|+K_3\sum_{j=1}^{p-1}|a_j-a_{j-1}|$$

$$\leq |a_0| + K_3[(|a_1| - |a_0|)\cos\alpha + (|a_1| + |a_0|)\sin\alpha + ...$$

$$+ ([a_{p-1}[- |a_{p-2}|)\cos\alpha + ([a_{p-1}[+ |a_{p-2}|)\sin\alpha \quad]$$

$$= |a_0| + K_3[\ (|a_{p-1}| - |a_0|)\cos\alpha + \sum (|a_j| + |a_{j-1}|)\sin\alpha \quad]$$

$$= |a_0| + K_3 m' \qquad (4)$$

Thus, for $|z| = K_3$, we have from (3) and (4), $|\psi(z)| < |\phi(z)|$.

Since $\phi(z)$ and $\psi(z)$ are analytic for $|z| \leq K_3$, it follows by Rouche's theorem that $\phi(z)$ and $\phi(z) + \psi(z)$ i.e. F(z) have the same number of zeros in $|z| < K_3$. But the zeros of P(z) are also the zeros of F(z). Therefore, we conclude that P(z) has at least p zeros in $|z| < K$, as the same is true of $\psi(z)$. That proves Theorem 2.

Proof of Theorem 3: Consider the function

$$F(z) = (z-1)f(z)$$

$$= (z-1)(a_0 + a_1 z + a_2 z^2 + ...)$$

$$= -a_0 + \sum_{j=1}^{p-1}(a_{j-1} - a_j)z^j + (a_{p-1} - a_p)z^p + \sum_{j=p+1}^{\infty}(a_{j-1} - a_j)z^j$$

$$= \phi(z) + \psi(z),$$

where

$$\phi(z) = -a_0 + \sum_{j=1}^{p-1}(a_{j-1} - a_j)z^j,$$

$$\psi(z) = (a_{p-1} - a_p)z^p + \sum_{j=p+1}^{\infty}(a_{j-1} - a_j)z^j.$$

For $|z| = K_4$ (K_4 <1, by hypothesis for $\frac{a_{p-1}}{a_p} < 2 + \frac{1}{p}$), we have

$$|\psi(z)| \geq |a_{p-1} - a_p|K_4^{\ p} - K_4^{\ p+1}\left(\sum_{j=p+1}^{\infty}|a_{j-1} - a_j|K_4^{\ j-(p+1)}\right)$$

$$\geq |a_{p-1} - a_p|K_4^{\ p} - K_4^{\ p+1}\left(\sum_{j=p+1}^{\infty}|a_{j-1} - a_j|\right)$$

$$= |a_{p-1} - a_p|K_4^{\ p} - K_4^{\ p+1}a_p$$

$$= |a_{p-1} - a_p|\left[(\frac{p}{p+1})(\frac{a_{p-1} - a_p}{a_p})\right] - \left[(\frac{p}{p+1})(\frac{a_{p-1} - a_p}{a_p})\right]a_p$$

$$= \left(\frac{p}{a_p}\right)^p\left(\frac{a_{p-1} - a_p}{p+1}\right)^{p+1}. \qquad (5)$$

and

$$|\phi(z)| \leq |a_0| + \sum_{j=1}^{p-1}|a_{j-1} - a_j|K_4^{\ j}$$

$$< |a_0| + [\ |a_0 - a_1| + |a_1 - a_2| + ... + |a_{p-2} - a_{p-1}|\ ]K_4$$

$$= |a_0| + [\ |\rho + a_0 - a_1 - \rho| + |a_1 - a_2| + ... + |a_{p-2} - a_{p-1}|\ ]K_4$$

$$\leq |a_0| + [\ |\rho + a_0 - a_1| + \rho + |a_1 - a_2| + ... + |a_{p-2} - a_{p-1}|\ ]K_4$$

$$= |a_0| + [\ \rho + a_0 - a_1 + \rho + a_1 - a_2 + ... + a_{p-2} - a_{p-1}\ ]K_4$$

$$= |a_0| + [\ 2\rho + a_0 - a_{p-1}\ ]K_4$$

$$= |a_0| + \frac{p}{a_p}\left(\frac{a_{p-1} - a_p}{p+1}\right)(2\rho + a_0 - a_{p-1}) \qquad (6)$$

Since, by hypothesis,

$$|a_0| + \frac{p}{a_p}\left(\frac{a_{p-1} - a_p}{p+1}\right)(2\rho + a_0 - a_{p-1}) < \left(\frac{p}{a_p}\right)^p \left(\frac{a_p - a_{p-1}}{p+1}\right)^{p+1},$$

it follows from (5) and (6) that $|\psi(z)| < |\phi(z)|$ for $|z| = K_4$.

Since $\phi(z)$ and $\psi(z)$ are analytic for $|z| \leq K_3$, it follows, by Rouche's theorem, that $\phi(z)$ and $\phi(z) + \psi(z)$ i.e. F(z) have the same number of zeros in $|z| < K_3$. But the zeros of P(z) are also the zeros of F(z). Therefore, we conclude that P(z) has at least p zeros in $|z| < K$, as the same is true of $\psi(z)$. That proves Theorem 3.

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