An Appropriate F -Test for Two-Way Balanced Interactive Model

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Abstract

The presence of interaction in a set of data/ model in a two-way interactive model may lead to a biased result when testing for the main effects. The nuisance parameter which is the interaction was removed from the data without distorting the assumption of homogeneity condition of analysis of variance. This is done by a linear combination such that the differences between the corresponding yield row-wise as well as column-wise difference is a constant and yet the total sum of the yield remains unchanged.

Keywords: Nuisance parameter, Mixed effect model, Least squares method.

1. Introduction

A major set back in design and analysis of experiments is the presence of interaction. A two-way factor interaction may be defined as the change in response due to one factor at different levels of the factor or two independent variables interact if the effects of one of the variables differ depending on the level of the other variable. The presence of interaction may obscure the result for the test of significance for the main effects according to Moore, et al [9]. One may be tempted to vehemently maintain that we should not even test for main effects once we know that interactions are present. Presence of interaction between the two factors means both effects are not independent. In analysis of variance, a large F -value provides evidence against the null hypothesis. However, the interaction test should be examined first. The reason for this is that, there is little point in testing the null hypothesis H_A of the alternative H_B if H_{AB} : no interaction effect is rejected, since the difference between any two levels of a main effect also includes an average interaction effect Cabrera and McDougall [2] argued.Overton [12] gave a quick check for interaction using an example on two binary factors A and B as illustrated below.A simple setting in which interactions can arise is in a two-factor experiment analyzed using Analysis of Variance (ANOVA) can be shown in Table 1. Suppose we have two binary factors A and B. For example, these factors might indicate whether either of two treatments was administered to a patient, with the treatments applied either singly, or in combination.

We can then consider the average treatment response (e.g. the symptom levels following treatment) for each patient, as a function of the treatment combination that was administered Overton [10] argued.

	Factor B	
Factor A	B_0	B_1
A_0	6	7
A_1	4	5

Table 1: Two-way classification without interaction

In Table 1, there is no interaction between the two treatments. Their effects are additive. The reason for this is that the difference in mean response between those subjects receiving treatment A and those not receiving treatment A is -2 regardless of whether treatment B is administered (4 - 6 = -2) or not (5 - 7 = -2). It automatically follows that the difference in mean response between those subjects receiving treatment B and those not receiving treatment B is the same regardless of whether treatment A is administered (7 - 6 = 5 - 4).

	Factor B	
Factor A	B_0	B_1
A_0	1	4
A_1	7	6

Table 2: Two-way classification with interaction

In contrast, in the Table 2 there is an interaction between the treatments and hence their effects are not additive.
Eze et al [3] developed a common F-test denominator for two-way interactive balanced design. In their work, they removed the interaction from the data and consequently divided the original data by the inverse of the square root of the standard error.Muhammad et al [11] argued that the effect of a medication is the sum of its drug effect placebo effect (meaning response) and their possible interaction. According to them, current interpretation of clinical trials' results assumes no interaction. Demonstrating such an interaction has been difficult due to lack of an appropriate design. However, this paper is set to resolve such problem.Park et al [13] carried out a research on the presence of interaction between direct and carry-over treatment effects by using a model in which the residual effect from a treatment depends upon the treatment applied in the succeeding period. This means a model which includes interaction between the treatment direct and residual effects. They assume that the residual effect do not persist further than one succeeding period.In the presence of higher order interaction, Kherad and Sara [7] demonstrated an exact permutation strategy applicable to fixed effect analysis of variance which can be used to test any factor.James [5] presented a paper similar to Overton [12]. In his paper, he demonstrated some common errors in interpreting interaction effects and the appropriate strategies for achieving post hoc understanding of the origin of detected interaction effects. According to him, a lack of interaction is often signified by parallel lines in a plot of cell means. Conversely, if the lines are not parallel, it signifies the presence of interaction.In ANOVA, a large F-value provides evidence against the null hypothesis. However, the interaction test should be examined first. The reason for this is that, there is little point in testing the null hypothesis H_A or H_B if H_{AB}: no interaction effect is rejected, since the difference between any two levels of a main effect also includes an average interaction Cabrera and MacDougall [2] argued.Moore et al [9] argued that there are three hypotheses in a two-way ANOVA with an F-test for each. We can test for significance of the main effects A, the main effect B and AB interaction. It is generally a good practice to examine the test interaction first, since the presence of strong interaction may influence the interpretation of the main effects.Some statistical softwares such as XLSTAT, SPSS, Minitab etc can perform the ANOVA test but does not consider the implications of the presence of interactions.

2. Methodology

Given the model for Two-way balanced interactive model

$$y_{ijk} = \mu + \alpha_i + \beta_j + \lambda_{ij} + e_{ijk} \begin{cases} i = 1, 2, \ldots, p \\ j = 1, 2, \ldots, q \\ k = 1, 2, \ldots, r \end{cases} \quad (1)$$

where

y_{ijk} is the kth observation in the ijth cell,

μ is a constant,

α_i is the average effects of factor A,

β_j is the average effects of factor B,

λ_{ij} is the interaction effect that exists between factor A and factor B and

e_{ijk} is the error associated with y_{ijk}.

The least square estimates of the parameters can be shown to be

$$\hat{\alpha}_i = \bar{y}_{i..} - \bar{y}_{...}$$

$$\hat{\beta}_j = \bar{y}_{.j.} - \bar{y}_{...}$$

$$\hat{\lambda}_{ij} = \bar{y}_{ij.} - \bar{y}_{i..} - \bar{y}_{.j.} + \bar{y}_{...}$$

From Table 1, the least square estimates of the cell observations have been calculated and presented in Table 3.

Factor A	Factor B B_0	B_1
A_0	5.5 + 1.0 - 0.5 + 0.0	5.5 + 1.0 - 0.5 + 0.0
A_1	5.5 - 1.0 - 0.5 + 0.0	5.5 - 1.0 + 0.5 + 0.0

Table 3: Least square estimates of Two-way classification without interaction

From Table 3, the interaction effects are zero.
In contrast, there are presences of interaction effects in Table 2 as shown in Table 4

Factor A	Factor B	
	B_0	B_1
A_0	4.5 - 2.0 - 0.5 -1.0	4.5 - 2.0 + 0.5 +1.0
A_1	4.5 - 2.0 - 0.5 -1.0	4.5 - 2.0 + 0.5 -1.0

Table 4: Least square estimates of Two-way classification with interaction

2.1 Expected mean squares

In this paper, Brute force method is used in deriving the expected mean squares. Brute force is a trial and error method used by application program to decode encrypted data rather than employing intellectual strategies (Bernstein [1]).Using the Brute force method the expected mean squares for Equation 1 has been derived and presented in Table 5.

S.V	d.f	SS	MS	All effects fixed	All effects random	Factor A fixed & factor B random	Factor A random & factor B fixed
Factor A	p-1	SS_α	MS_α	$qr\sum_i \frac{\alpha_i^2}{p-1}+\sigma_e^2$	$qr\sigma_\alpha^2 + r\sigma_\lambda^2+\sigma_e^2$	$qr\sum_i \frac{\alpha_i^2}{p-1}+ r\sigma_\lambda^2+\sigma_e^2$	$r\sigma_\alpha^2+\sigma_e^2$
Factor B	q-1	SS_β	MS_β	$pr\sum_j \frac{\beta_j^2}{q-1}+\sigma_e^2$	$pr\sigma_\beta^2 + r\sigma_\lambda^2+\sigma_e^2$	$pr\sigma_\beta^2+\sigma_e^2$	$pr\sum_j \frac{\beta_j^2}{q-1}+ r\sigma_\lambda^2+\sigma_e^2$
AB interaction	(p-1)(q-1)	SS_λ	MS_λ	$r\sum_{ij} \frac{\lambda_{ij}^2}{(p-1)(q-1)}+\sigma_e^2$	$r\sigma_\lambda^2+\sigma_e^2$	$r\sigma_\lambda^2+\sigma_e^2$	$r\sigma_\lambda^2+\sigma_e^2$
Error	pq(r-1)	SS_e	MS_e	σ_e^2	σ_e^2	σ_e^2	σ_e^2
Total	pqr-1	SS_T		-	-	-	-

Table 5: Complete ANOVA Table

From Table 5, there is no obvious denominator for testing for the main effects when the model/data are fixed, random or mixed. For instance, if the data are fixed, the common denominator for testing for the main effects is MS_e. Similarly, if the data were random, the common denominator for testing for the main effects is MS_λ under the null hypothesis H_0. When the data are mixed, the denominator of the F -ratio varies. The reason for this is the presence of interaction. If the interaction is removed from the data, Table 5 reduces to Table 6.

S.V	d.f	SS	MS	All effects fixed	All effects random	Factor A fixed & factor B random	Factor A random & factor B fixed
Factor A	p-1	SS_α	MS_α	$qr\sum_i \frac{\alpha_i^2}{p-1}+\sigma_e^2$	$qr\sigma_\alpha^2+\sigma_e^2$	$qr\sum_i \frac{\alpha_i^2}{p-1}+\sigma_e^2$	$r\sigma_\alpha^2+\sigma_e^2$
Factor B	q-1	SS_β	MS_β	$pr\sum_j \frac{\beta_j^2}{q-1}+\sigma_e^2$	$pr\sigma_\beta^2+\sigma_e^2$	$pr\sigma_\beta^2+\sigma_e^2$	$pr\sum_j \frac{\beta_j^2}{q-1}+ \sigma_e^2$
Error	(p-1)(q-1)	SS_e	MS_e	σ_e^2	σ_e^2	σ_e^2	σ_e^2
Total	pqr-1	SS_T		-	-	-	-

Table 6: Reduced ANOVA Table

From Table 6, the common denominator for testing for the main effects is MS_e under H_0 and the model equation consequently reduces to Equation 2.

$$y_{ijk} = \mu + \alpha_i + \beta_j + e_{ijk} \begin{cases} i = 1, 2, \ldots, p \\ j = 1, 2, \ldots, q \\ k = 1, 2, \ldots, r \end{cases} \tag{2}$$

The parameters have the same meaning as it is in Equation 1.

2.2 Method of removing the interaction

The interactions can be removed from the data/model as follows without distorting the assumptions of analysis of variance.Any linear combination such that the differences between the corresponding yield row-wise as well as column-wise differences is a constant and yet the total sum of the yield remains unchanged eliminates the interaction (Weisstein[15]).Let $x_{11}, x_{12}, \ldots, x_{pq}$ be the yields or values in Two-way crossed interactive model with one observation per cell. The data format is shown in Table 7.

Factor A	Factor B 1	2	3	. . ., q	$\bar{y}_{i.}$	$T_{i.}$
1	y_{11}	y_{12}	y_{13}	y_{1q}	$\bar{y}_{1.}$	$T_{1.}$
2	y_{21}	y_{22}	y_{23}	y_{2q}	$\bar{y}_{2.}$	$T_{2.}$
3	y_{31}	y_{32}	y_{33}	y_{3q}	$\bar{y}_{3.}$	$T_{3.}$
.	.	.	.	.	.	.
.	.	.	.	.	.	.
.	.	.	.	.	.	.
p	y_{p1}	y_{p2}	y_{p3}	y_{pq}	$\bar{y}_{p.}$	
$T_{.j}$	$T_{.1}$	$T_{.2}$	$T_{.3}$	$T_{.q}$		$T_{..}$

Table 7: Data layout of Two-way classification with one observation per cell.

From Table 7

$$y_{12} - y_{11} = k \Rightarrow y_{12} = k + y_{11}$$
$$y_{13} - y_{12} = k \Rightarrow y_{13} = k + y_{12} = 2k + y_{11}$$
$$y_{21} - y_{13} = k \Rightarrow y_{21} = k + y_{13} = 3k + y_{11}$$
$$y_{22} - y_{21} = k \Rightarrow y_{22} = k + y_{21} = 4k + y_{11}$$
$$y_{ij} - y_{i'j'} = k \Rightarrow y_{ij} = k + y_{i'j'} = (pq-1)k + y_{11} \tag{3}$$
$$ij \neq i'j'$$

3. Illustrative Example

An engineer is designing a battery for use in a device that will be subjected to some extreme variation in temperature. The only design parameter that he can select is the plate material for the battery, and he has three possible choices.The engineer decides to test all the three materials at three temperature levels- $15^0 F, 70^0 F$, and $125^0 F$ -as these temperature levels are consistent with the product end-use environment. Four batteries are tested at each combination of plate material and temperature; all 36 tests are run in a random. The experiment and the resulting observed battery life data are given in Table 8.

Material type	Temperature (^{0}F) 15	70	125	$T_{..k}$
1	130, 155, 74, 180	34, 40, 80, 75	20, 70, 82, 58	
2	150, 188, 159, 126	136, 122, 106, 115	25, 70, 58, 45	
3	138, 110, 168, 160	174, 120, 150, 139	96, 104, 82, 60	
				$T_{..1} = 903$
				$T_{..2} = 979$
				$T_{..3} = 959$
				$T_{..4} = 958$
				$T_{...} = 3799$

Table 8: Source: Life Data for battery design from [10] p.207

Using SPSS, the analysis of variance was performed and presented in Table 9.

Source	Type 111 sum of squares	df	Mean Square	F	Sig
Corrected model	59416.22	8	7427.028	11.000	0.000
Intercept	400900.028	1	400900.028	593.739	0.000
Material	10683.722	2	5341.861	7.911	0.002
Temperature	39118.722	2	19559.361	28.968	0.000
Material*Temperature	9613.778	4	2403.44	3.560	0.019
Error	18230.750	27	675.216		
Total	478547.00	36			
Corrected Total	77646.972	35			

Table 9: ANOVA Table

From the ANOVA Table 9, the main effects and the interaction are significant.
Using the expression derived from Table 7 and Equation (3), the data in Table 8 are now transformed as follows:
For the corresponding entries for y_{11} we have

$$36k - 1170 = 903 \Rightarrow k = -7.42$$

The values of k for the corresponding entries for y_{112}, y_{113}, and y_{114} are -11.56, 8.14, and -18.39 respectively.

The transformed data are shown in Table 10.

Material type	Temperature (^{0}F) 15	70	125	$T_{..k}$
1	130, 155, 74, 180	122.58, 143.44, 82.14, 161.61	115.16, 131.88, 90.28, 143.22	
2	107.74, 120.32, 98.42, 124.83	100.32, 108.76, 106.56, 106.44	92.90, 97.20, 114.7, 88.05	
3	85.48, 85.64, 122.84, 69.66	78.06, 74.08, 130.98, 51.27	70.64, 62.52, 139.12, 32.88	
				$T_{..1} = 902.88$
				$T_{..2} = 978.84$
				$T_{..3} = 959.04$
				$T_{..4} = 957.96$
				$T_{...} = 3798.72$

Table 10: Transformed data of the life Data for battery design

The ANOVA test for the transformed data is shown in Table 11

Source	Type 111 sum of squares	df	Mean Square	*F*	Sig
Corrected model	12815.894	8	1601.987	1.870	0.1070
Intercept	400840.934	1	400840.934	467.952	0.000
Material	11534.304	2	5767.152	6.733	0.004
Temperature	1281.589	2	640.795	28.9680.748	0.48
Material*Temperature	0.000	4	0.000	0.000	1.000
Error	23127.7999	27	856.585		
Total	436784.627	36			
Corrected Total	35943.692	35			

Table 11: ANOVA Table

From Table 11, the interaction between the material and temperature is highly non-significant showing the successful removal of the interaction from the data. However, the material effects are significant while the temperature effects are non-significant.Since the interactions effects are zero, we therefore perform the ANOVA test in absence of the interaction as shown in Table 12.

Source	Type 111 sum of squares	df	Mean Square	*F*	Sig
Corrected model	12815.894	4	3203.973	4.295	0.007
Intercept	400840.934	1	400840.934	537.278	0.000
Material	11534.304	2	5767.152	7.730	0.002
Temperature	1281.589	2	640.795	0.859	0.433
Error	23127.7999	31	746.058		
Total	436784.627	36			
Corrected Total	35943.692	35			

Table 12: Reduced ANOVA Table

The results obtained in Table 12 are the same as the result obtained in Table 11.
If we are to use the method of 3.0 we shall have the same transformed data in Table 10 and ANOVA test in Table 12.

4. Summary And Conclusion

We have successfully derived an expression that would enable us remove the interaction in our data/model. From the illustrative example given, is possible to commit an error when interaction is present in our data. For example, the temperature effects were significant when interaction is present as shown in Table 9. When the interaction effects were removed from the data, the temperature effects became non-significant.
We therefore recommend that analysis of variance test should be done when interaction is highly non-significant or zero.

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