Polynomials having no Zero in a Given Region

M. H. Gulzar
Department of Mathematics University of Kashmir, Srinagar 190006

ABSTRACT:
In this paper we consider some polynomials having no zeros in a given region. Our results when combined with some known results give ring–shaped regions containing a specific number of zeros of the polynomial.

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I. INTRODUCTION AND STATEMENT OF RESULTS

In the literature we find a large number of published research papers concerning the number of zeros of a polynomial in a given circle. For the class of polynomials with real coefficients, Q. G. Mohammad [5] proved the following result:

Theorem A: Let \( P(z) = \sum_{j=0}^{\infty} a_j z^j \) be a polynomial of degree \( n \) such that

\[
a_n \geq a_{n-1} \geq \ldots \geq a_1 \geq a_0 > 0.
\]

Then the number of zeros of \( P(z) \) in \( |z| \leq \frac{1}{2} \) does not exceed

\[
1 + \frac{1}{\log 2} \log \frac{a_n}{a_0}.
\]

Bidkham and Dewan [1] generalized Theorem A in the following way:

Theorem B: Let \( P(z) = \sum_{j=0}^{\infty} a_j z^j \) be a polynomial of degree \( n \) such that

\[
a_n \leq a_{n-1} \leq \ldots \leq a_{k+1} \leq a_k \geq a_{k-1} \geq \ldots \leq a_1 \geq a_0 > 0,
\]

for some \( k, 0 \leq k \leq n \). Then the number of zeros of \( P(z) \) in \( |z| \leq \frac{1}{2} \) does not exceed

\[
\frac{1}{\log 2} \log \left( \frac{|a_n| + |a_0| - a_n - a_0 + 2a_k}{|a_k|} \right).
\]

Ebadian et al [2] generalized the above results by proving the following results:

Theorem C: Let \( P(z) = \sum_{j=0}^{\infty} a_j z^j \) be a polynomial of degree \( n \) such that

\[
a_n \leq a_{n-1} \leq \ldots \leq a_{k+1} \leq a_k \geq a_{k-1} \geq \ldots \geq a_0
\]

for some \( k, 0 \leq k \leq n \). Then the number of zeros of \( P(z) \) in \( |z| \leq \frac{R}{2} \), \( R > 0 \), does not exceed

\[
\frac{1}{\log 2} \log \left( \frac{|a_n| R^{n+1} + |a_0| + R^k (a_k - a_0) + R^k (a_k - a_n)}{|a_k|} \right)
\]

for \( R \geq 1 \).
and
\[
\frac{1}{\log c} \log \left\{ \frac{|a_n| R^{n+1} + |a_o| + R(a_{a-} - a_o) + R^\kappa(a_{a-} - a_o) + R^\kappa(a_{a-} - a_o)}{|a_n|} \right\} \quad \text{for } R \leq 1.
\]

M.H. Gulzar [3] generalized the above result by proving the following result:

**Theorem D:** Let \( P(z) = \sum_{j=0}^{n} a_j z^j \) be a polynomial of degree \( n \) with \( \text{Re}(a_j) = \alpha_j, \ \text{Im}(a_j) = \beta_j \) such that for some \( k, \tau, , 0 < k \leq 1, 0 < \tau \leq 1, 0 \leq \lambda \leq n, \)
\[ k \alpha_n \leq \alpha_{n-1} \leq \ldots \leq \alpha_{\lambda+1} \leq \alpha_{\lambda} \geq \alpha_{\lambda-1} \geq \ldots \geq \tau \alpha_0. \]

Then the number of zeros of \( P(z) \) in \( |z| \leq \frac{R}{c} (R > 0, c > 1) \) does not exceed
\[
\frac{1}{\log c} \log \left\{ \frac{|a_n| R^{n+1} + |a_o| + R^\kappa[\alpha_{\lambda} - \tau(\alpha_o + \alpha_n) + |\beta_n| + |\beta_\lambda| + 2 \sum_{j=1}^{\lambda-1} |\beta_j|]}{|a_n|} \right\}
\]
for \( R \geq 1 \)
\[
\frac{1}{\log c} \log \left\{ \frac{|a_n| R^{n+1} + |a_o| + R^\kappa[\alpha_{\lambda} - \tau(\alpha_o + \alpha_n) + |\beta_n| + |\beta_\lambda| + 2 \sum_{j=1}^{\lambda-1} |\beta_j|]}{|a_n|} \right\}
\]
and

In this paper we prove the following result:

**Theorem 1:** Let \( P(z) = \sum_{j=0}^{n} a_j z^j \) be a polynomial of degree \( n \) with \( \text{Re}(a_j) = \alpha_j, \ \text{Im}(a_j) = \beta_j \) such that for some \( k, \tau, , 0 < k \leq 1, 0 < \tau \leq 1, 0 \leq \lambda \leq n, \)
\[ k \alpha_n \leq \alpha_{n-1} \leq \ldots \leq \alpha_{\lambda+1} \leq \alpha_{\lambda} \geq \alpha_{\lambda-1} \geq \ldots \geq \tau \alpha_0. \]

Then \( P(z) \) has no zero in \( |z| < \frac{|a_i|}{M_1} \) for \( R \geq 1 \) and no zero in \( |z| < \frac{|a_i|}{M_2} \) for \( R \leq 1 \) where
\[
M_1 = |a_n| R^{n+1} + R^\kappa[\alpha_{\lambda} - \tau(\alpha_o + \alpha_n) + |\beta_n| + |\beta_\lambda| + 2 \sum_{j=\lambda+1}^{\lambda-1} |\beta_j|] + R^\kappa[\alpha_{\lambda} - \tau(\alpha_o + \alpha_n) + |\beta_n| + |\beta_\lambda| + 2 \sum_{j=1}^{\lambda-1} |\beta_j|]
\]
and
\[
M_2 = |a_n| R^{n+1} + R^\kappa[\alpha_{\lambda} - \tau(\alpha_o + \alpha_n) + |\beta_n| + |\beta_\lambda| + 2 \sum_{j=\lambda+1}^{\lambda-1} |\beta_j|] + R[\alpha_{\lambda} - \tau(\alpha_o + \alpha_n) + |\beta_n| + |\beta_\lambda| + 2 \sum_{j=1}^{\lambda-1} |\beta_j|].
\]
Combining Theorem 1 with Theorem D, we get the following result:

**Theorem 2:** Let \( P(z) = \sum_{j=0}^{\infty} a_j z^j \) be a polynomial of degree \( n \) with \( \text{Re}(a_j) = \alpha_j, \text{Im}(a_j) = \beta_j \) such that for some \( k, \tau, \lambda, 0 < k \leq 1, 0 < \tau \leq 1, 0 \leq \lambda \leq n \),

\[
 k\alpha_n \leq \alpha_{n-1} \leq \ldots \leq \alpha_{\lambda+1} \leq \alpha_{\lambda} \geq \alpha_{\lambda-1} \geq \ldots \geq \alpha_0.
\]

Then the number of zeros of \( P(z) \) in \( \frac{|a_0|}{M_1} \leq |z| \leq \frac{R}{c} \) \((R > 0, c > 1)\) does not exceed

\[
 \frac{1}{\log c} \log \left\{ \frac{|a_n| R^{n+1} + |a_0| + R^\lambda [\alpha_\lambda - \tau (|\alpha_0| + \alpha_\lambda) + |\beta_0| + |\beta_\lambda| + 2 \sum_{j=1}^{\lambda} |\beta_j|]}{|a_0|} \right\}
\]

for \( R \geq 1 \)

and the number of zeros of \( P(z) \) in \( \frac{|a_0|}{M_2} \leq |z| \leq \frac{R}{c} \) \((R > 0, c > 1)\) does not exceed

\[
 \frac{1}{\log c} \log \left\{ \frac{|a_n| R^{n+1} + |a_0| + R^{\lambda} [\alpha_\lambda - \tau (|\alpha_0| + \alpha_\lambda) + |\beta_0| + |\beta_\lambda| + 2 \sum_{j=1}^{\lambda} |\beta_j|]}{|a_0|} \right\}
\]

for \( R \leq 1 \),

where \( M_1 \) and \( M_2 \) are as given in Theorem 1.

For different values of the parameters, we get many interesting results including some already known results.

### 2. Proofs of Theorems

**Proof of Theorem 1:** Consider the polynomial

\[
 F(z) = (1-z)P(z) = (1-z)(a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0)
\]

\[
 = -a_n z^{n+1} + (a_n - a_{n-1}) z^n + \ldots + (a_1 - a_0) z + a_0
\]

\[
 = -a_n z^{n+1} + (k \alpha_n - \alpha_{n-1}) z^n + \sum_{j=\lambda+1}^{n} (\alpha_j - \alpha_{j-1}) z^j + \sum_{j=2}^{\lambda} (\alpha_j - \alpha_{j-1}) z^j + \sum_{j=2}^{\lambda} (\alpha_1 - \alpha_0) z + \sum_{j=2}^{\lambda} (\alpha_1 - \alpha_0) z + \sum_{j=2}^{\lambda} (\alpha_1 - \alpha_0) z
\]

\[
 = a_0 + G(z), \text{ where}
\]
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\[ G(z) = -a_n z^{n+1} + a_0 + [k(\alpha_n - \alpha_{n-1}) - (k - 1)\alpha_n]z^n + \sum_{j=1}^{n-1} (\alpha_j - \alpha_{j-1})z^j + \sum_{j=2}^{k} (\alpha_j - \alpha_{j-1})z^j + [k(\alpha_0 - \tau \alpha_n) + (\tau - 1)\alpha_0]z + i\sum_{j=1}^{k} (\beta_j - \beta_{j-1})z^j \]

For \(|z| \leq R\), we have, by using the hypothesis

\[
|G(z)| \leq |a_n| R^{n+1} + [(\alpha_{n-1} - k\alpha_n) + (1 - k)|\alpha_n|]R^n + \sum_{j=1}^{n-1} (\alpha_j - \alpha_{j-1})R^j + \sum_{j=2}^{k} (\alpha_j - \alpha_{j-1})R^j + [(\alpha_1 - \tau \alpha_0) + (1 - \tau)|\alpha_0|]R + \sum_{j=1}^{k} (|\beta_j| + |\beta_{j-1}|)R^j
\]

which gives

\[
|G(z)| \leq |a_n| R^{n+1} + R^n |\alpha_n| + |k(\alpha_n) + \alpha_n| + |\alpha_0| + |\beta_0| + |\beta_1| + 2 \sum_{j=1}^{n} |\beta_j|
\] + \[R^2 |\alpha_0 - \tau(\alpha_0 + \alpha_n) + |\alpha_1| + |\beta_1| + 2 \sum_{j=1}^{k-1} |\beta_j|
\]

\[= M_1 \quad \text{for } R \geq 1\]

and

\[
|G(z)| \leq |a_n| R^{n+1} + R^n |\alpha_n| + k(\alpha_n) + |\alpha_0| + |\beta_0| + |\beta_1| + 2 \sum_{j=1}^{n} |\beta_j|
\] + \[R^2 |\alpha_0 - \tau(\alpha_0 + \alpha_n) + |\alpha_1| + |\beta_1| + 2 \sum_{j=1}^{k-1} |\beta_j|
\]

\[= M_2 \quad \text{for } R \leq 1\]

Since \(G(z)\) is analytic in \(|z| \leq R\) and \(G(0)=0\), it follows by Schwarz Lemma that

\[|G(z)| \leq M_1 |z| \quad \text{for } R \geq 1 \quad \text{and} \quad |G(z)| \leq M_2 |z| \quad \text{for } R \leq 1\]

Hence, for \(R \geq 1\),

\[|F(z)| = |a_0 + G(z)| \geq |a_0| - |G(z)| \geq |a_0| - M_1 |z| \geq 0\]

if \(|z| < \frac{|a_0|}{M_1}\).

And for \(R \leq 1\),

\[|F(z)| = |a_0 + G(z)| \geq |a_0| - |G(z)| \geq |a_0| - M_2 |z| \geq 0\]

if \(|z| < \frac{|a_0|}{M_2}\).

This shows that \(F(z)\) has no zero in \(|z| < \frac{|a_0|}{M_1}\) for \(R \geq 1\) and no zero in \(|z| < \frac{|a_0|}{M_2}\) for \(R \leq 1\). But the zeros of \(P(z)\) are also the zeros of \(F(z)\). Therefore, the result follows.
REFERENCES


