Fundamental Theorem Of Algebra A Study

Dr Mushtaq Ahmad Shah
Department of Mathematics CMJ University Shilliong 79300

Abstract: This paper is a study of the Fundamental Theorem of Algebra which states that every polynomial equation of degree n has exactly n zeroes. It gives a historical account of the theorem in different periods; by different mathematicians it also includes the contribution of different countries. In addition to this I present different proofs of Fundamental Theorem of Algebra by using different techniques which is actually the main theme behind the paper.

Keywords:- Polynomials, zeroes, analytic, bounded, constant, Maximum Modulus,

1. Introduction

When we speak of the early history of algebra, first of all it is necessary to consider the meaning of the term. If by algebra we mean the science which allows us to solve the equation $ax^2 + bx + c = 0$, expressed in these symbols, then the history begins in the 17^{th} Century ; if we remove the restrictions as to these particular signs and allow for other and less convenient symbols. we might properly begins the history in the 3^{rd} century ; if we allow for the solution of the above equation by geometric methods, without algebraic symbols of any kind, we might say that the algebra begins with the Alexandrian School or a little earlier; and if we say we should class as algebra any problem that we should now solve by algebra (even through it was at first solved by mere guessing or by some cumbersome arithmetic process), then the science was known about I800 B.C,, and probably still earlier. It is first proposed to give a brief survey of the development of algebra, recalling the names of those who helped to set the problems that were later solved by the aid of equation, as well as those who assisted in establishing the science itself. These names have been mentioned in Volume 1 and some of them will be referred to when we consider the development of the special topics of algebra and their application to the solution of the elementary problems. It should be borne in mind that most ancient writers outside Greece included in their mathematics works a wide range of subjects. Ahmes (c.1550 B.C.), for example, combines his algebra with arithmetic and mensuration, and even shows some evidence that trigonometry was making a feeble start. There was no distinct treatise on algebra before the time of Diophantus (c.275). There are only four Hindu writers on algebra whose are particularly noteworthy. These are Aryabhata, whose Aryabha-tiyam(c.510) included problems in series, permutation , and linear and quadratic equations; Brahmagupta, whose Brahmasid-dhanta(c.628) contains a satisfactory rule for the solving the quadratic, and whose problems included the subjects treated by Aryabhata : Mahavira whose Ganita-Sari Sangraha (c.850) contains a large number of problems involving series, radicals, and equations; and Bhaskara, whose Bija Ganita (c.1150) contains nine chapters and extended the work through quadratic equations.

Algebra in the modern sense can hardly be said to have existed in the golden age of Greek mathematics. The Greeks of the classical period could solve many algebraic problems of considerable difficulty, but the solutions were all geometric. Hippocrates (c.460 B.C.), for example, assumed a construction which is equivalent to solving the equation

$$x^2 + \sqrt{\frac{3}{2}} \cdot ax = a^2 .$$

With Diophantus (c.275) there first enters an algebraic symbolism worthy of the name, and also a series of purely algebraic problems treated by analytic methods. Many of his equations being indeterminate, equation of this type are often called Diophantine Equations. His was the first work devoted chiefly to algebra, and on his account he is often, and with must justice, called the father of the science. The algebraists of special prominence among the Arabs and Persians were Mohammed ibn Musa, al- Khowarizmi, whose al- jabr w'al muqabalah(c.825) gave the name to the science and contained the first systematic treatment of the general subject as distinct from the theory of numbers; Almahani (c.860), whose name will be mentioned in connection with the cubic; Abu kamil (c.900), who drew extensively from al- khawarizmi and from whom Fibonacci (1202) drew in turn; al-Karkhi(c.1020), whose Fakhri contains various problems which still form part of the general stock material of algebra; and Omar Khanyyam (c.1100), whose algebra was the best that the Persian writers produced.

Most of the medieval Western scholars who helped in the progress of algebra were translators from the Arabic. Among these were Johannes Hispalensis (c.1140), who may have translated al-Khowarizmi's algebra; Gherardo of Cremona (c.1150), to whom is also attributed a translation of the same work; Adelard of Bath (c.1120), who probably translated an asronomical work of al-Khowarizmi, and who certainly helped to make this writer Known; and Robert of Chester, whose translation of al-Khoarizmi's algebra is now available in English.

The first epoch-making algebra to appear in print was the Ars Magna of Cardan (1545). This was devoted primarily to the solutions of algebraic equations. It contained the solutions of the cubic and biquadratic equations, made use of complex numbers, and in general may be said to have been the first step towards modern algebra.The next great work on algebra to appear in print was the General Trattato of Tartaglia (1556-1560), although his side of the controversy with Cardan over the solution of the cubic equation had already been given in his qvesitised inventioni diverse (1546). The first noteworthy attempt to write algebra in England was made by Robert Recorde, who Whetstone of Witte (1557) was an excellent textbook for its first time. The next important contribution was Masterson's incomplete treatise of 1592-1595, but the work was not up to standard set by Recorde. The first Italian textbook to bear the title of algebra was Bombelli's work of 1572. In this book the material is arranged with some attention to the teaching of the subject.

By this time elementary algebra was fairly well perfected and it only remained to develop a good symbolism. Every real polynomial can be expressed as the product of real linear and real quadratic factors.Early studies of equations by al-Khwarizmi (c 800) only allowed positive real roots and the Fundamental Theorem of Algebra was not relevant. Cardan was the first to realise that one could work with quantities more general than the real numbers. This discovery was made in the course of studying a formula which gave the roots of a cubic equation. The formula when applied to the equation x3 = 15x + 4 gave an answer involving $\sqrt{-121}$ yet Cardan knew that the equation had x = 4 as a solution. He was able to manipulate with his 'complex numbers' to obtain the right answer yet he in no way understood his own mathematics. Bombelli, in his Algebra, published in 1572, was to produce a proper set of rules for manipulating these 'complex numbers'. Descartes in 1637 says that one can 'imagine' for every equation of degree n, n roots but these imagined roots do not correspond to any real quantity Viète gave equations of degree n with n roots but the first claim that there are always n solutions was made by a Flemish mathematician Albert Girard in 1629 in L'invention en algebre However he does not assert that solutions are of the form a + bi, a, b real, so allows the possibility that solutions come from a larger number field than C. In fact this was to become the whole problem of the Fundamental Theorem of Algebra for many years since mathematicians accepted Albert Girard's assertion as self-evident. They believed that a polynomial equation of degree n must have n roots, the problem was, they believed, to show that these roots were of the form a + bi, a, b real Now Harriot knew that a polynomial which vanishes at t has a root x - t but this did not become well known until stated by Descartes in 1637 in La geometrie, so Albert Girard did not have much of the background to understand the problem properly.

A 'proof' that the Fundamental Theorem of Algebra was false was given by Leibniz in 1702 when he asserted that $x^4 + t^4$ could never be written as a product of two real quadratic factors. His mistake came in not realizing that $\sqrt{i}$ could be written in the form a + bi, a, b real. Euler, in a 1742 correspondence with Nicolaus(II) Bernoulli and Goldbach, showed that the Leibniz counter example was false. D'Alembert in 1746 made the first serious attempt at a proof of the Fundamental Theorem of Algebra . For a polynomial f(x) he takes a real b, c so that f (b) = c. Now he shows that there are complex numbers z1 and w_1 so that

$$|z_1| < |c|, |w_1| < |c|.$$

He then iterates the process to converge on a zero of f. His proof has several weaknesses. Firstly, he uses a lemma without proof which was proved in 1851 by Puiseau, but whose proof uses the Fundamental Theorem of Algebra Secondly, he did not have the necessary knowledge to use a compactness argument to give the final convergence. Despite this, the ideas in this proof are important. Euler was soon able to prove that every real polynomial of degree n, $n \leq 6$ had exactly n complex roots. In 1749 he attempted a proof of the general case, so he tried to prove the Fundamental Theorem of Algebra for Real Polynomials: Every polynomial of the nth degree with real coefficients has precisely n zeros in C. His proof in Recherches sur les racines imaginaires des équations is based on decomposing a monic polynomial of degree 2^n into the product of two monic polynomials of degree $m = 2^{n-1}$. Then since an arbitrary polynomial can be converted to a monic polynomial by multiplying by ax^k for some k the theorem would follow by iterating the decomposition. Now Euler knew a fact which went back to Cardan in Ars Magna, or earlier, that a transformation could be applied to remove the second largest degree term of a polynomial. Hence he assumed that

$$\begin{aligned} x^{2m} + Ax^{2m-2} + Bx^{2m-3} + \ldots = (x^m + tx^{m-1} + gx^{m-2} \\ + \ldots)(x^m - tx^{m-1} + hx^{m-2} + \ldots) \end{aligned}$$

and then multiplied up and compared coefficients. This Euler claimed led to g, h, ... being rational functions of A, B, ..., t. All this was carried out in detail for n = 4, but the general case is only a sketch. In 1772 Lagrange raised objections to Euler's proof. He objected that Euler's rational functions could lead to 0/0. Lagrange used his knowledge of permutations of roots to fill all the gaps in Euler's proof except that he was still assuming that the polynomial equation of degree n must have n roots of some kind so he could work with them and deduce properties, like eventually that they had the form a + bi, a, b real. Laplace, in 1795, tried to prove the Fundamental Theorem of Algebra using a completely different approach using the discriminant of a polynomial. His proof was very elegant and its only 'problem' was that again the existence of roots was assumed Gauss is usually credited with the first proof of the Fundamental Theorem of Algebra . In his doctoral thesis of 1799 he presented his first proof and also his objections to the other proofs. He is undoubtedly the first to spot the fundamental flaw in the earlier proofs, to which we have referred many times above, namely the fact that they were assuming the existence of roots and then trying to deduce properties of them. Of Euler's proof Gauss says ... if one carries out operations with these impossible roots, as though they really existed, and says for example, the sum of all roots of the equation

$$x^m+ax^{m-1}+bx^{m-2}+...=0$$

is equal to –a even though some of them may be impossible (which really means: even if some are non-existent and therefore missing), then I can only say that I thoroughly disapprove of this type of argument. Gauss himself does not claim to give the first proper proof. He merely calls his proof new but says, for example of d'Alembert's proof, that despite his objections a rigorous proof could be constructed on the same basis. Gauss's proof of 1799 is topological in nature and has some rather serious gaps. It does not meet our present day standards required for a rigorous proof. In 1814 the Swiss accountant Jean Robert Argand published a proof of the Fundamental Theorem of Algebra which may be the simplest of all the proofs. His proof is based on d'Alembert's 1746 idea. Argand had already sketched the idea in a paper published two years earlier Essai sur une manière de représenter les quantitiés imaginaires dans les constructions géometriques. In this paper he interpreted i as a rotation of the plane through 90° so giving rise to the Argand plane or Argand diagram as a geometrical representation of complex numbers. Now in the later paper Réflexions sur la nouvelle théorie d'analyse Argand simplifies d'Alembert's idea using a general theorem on the existence of a minimum of a continuous function.

In 1820 Cauchy was to devote a whole chapter of Cours d'analyse to Argand's proof (although it will come as no surprise to anyone who has studied Cauchy's work to learn that he fails to mention Argand !) This proof only fails to be rigorous because the general concept of a lower bound had not been developed at that time. The Argand proof was to attain fame when it was given by Chrystal in his Algebra textbook in 1886. Chrystal's book was very influential.Two years after Argand's proof appeared Gauss published in 1816 a second proof of the Fundamental Theorem of Algebra . Gauss uses Euler's approach but instead of operating with roots which may not exist, Gauss operates with indeterminates. This proof is complete and correct. A third proof by Gauss also in 1816 is, like the first, topological in nature. Gauss introduced in 1831 the term 'complex number'. The term 'conjugate' had been introduced by Cauchy in 1821. Gauss's criticisms of the Lagrange-Laplace proofs did not seem to find immediate favour in France. Lagrange's 1808 2nd Edition of his treatise on equations makes no mention of Gauss's new proof or criticisms. Even the 1828 Edition, edited by Poinsot, still expresses complete satisfaction with the Lagrange- Laplace proofs and no mention of the Gauss criticisms. In 1849 (on the 50th anniversary of his first proof!) Gauss produced the first proof that a polynomial equation of degree n with complex coefficients has n complex roots. The proof is similar to the first proof given by Gauss. However little since it is straightforward to deduce the result for complex coefficients from the result about polynomials with real coefficients.

It is worth noting that despite Gauss's insistence that one could not assume the existence of roots which were then to be proved reals he did believe, as did everyone at that time, that there existed a whole hierarchy of imaginary quantities of which complex numbers were the simplest. Gauss called them a shadow of shadows. t was in searching for such generalisations of the complex numbers that Hamilton discovered the quaternions around 1843, but of course the quaternions are not a commutative system. The first proof that the only commutative algebraic field containing R was given by Weierstrass in his lectures of 1863. It was published in Hankel's book Theorie der complexen Zahlensysteme.Of course the proofs described above all become valid once one has the modern result that there is a splitting field for every polynomial. Frobenius, at the celebrations in Basle for the bicentenary of Euler's birth said Euler gave the most algebraic of the proofs of the existence of the roots of an equation, the one which is based on the proposition that every real equation of odd degree has a real root. I regard it as unjust to ascribe this proof exclusively to Gauss, who merely added the finishing touches. The Argand proof is only an existence proof and it does not in any way allow the roots to be constructed. Weierstrass noted in 1859 made a start towards a constructive proof but it was not until 1940 that a constructive variant of the Argand proof was given by Hellmuth Kneser. This proof was further simplified in 1981 by Martin Kneser, Hellmuth Kneser's son. In this dissertation we shall use various analytical approaches to prove the theorem All proofs below involve some analysis, at the very least the concept of continuity of real or complex functions. Some also use differentiable or even analytic functions.

This fact has led some to remark that the Fundamental Theorem of Algebra is neither fundamental, nor a theorem of algebra.
Some proofs of the theorem only prove that any non-constant polynomial with real coefficients has some complex root. This is enough to establish the theorem in the general case because, given a non-constant polynomial *p*(*z*) with complex coefficients, the polynomial

$$q(z) = p(z)\overline{p(\overline{z})}$$

has only real coefficients and, if *z* is a zero of *q*(*z*), then either *z* or its conjugate is a root of *p*(*z*).

Different Proofs of the theorem: Statement of Fundamental theorem of algebra Every polynomial equation of degree n has exactly n zeroes An expression of the form
$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$
Where $a_0, a_1 \ . \ . \ . \ a_{n-1}, a_n \neq 0$

are real or complex numbers and p(x) is called a polynomial equation of degree n and the equation p(x) = 0 is called a polynomial equation of degree n .
By a zero of the polynomial x or a root of the equation x = 0 , we mean a value of p(x) such that p(x) = 0.

First proof: For the proof of the Theorem we must know the following Theorem known as Liouville's Theorem.

STATEMENT: If a function f(z) is analytic for all finite values of z and is bounded, then f(z) is constant. "or"

If f is regular in whole z-plane and if $|f(z)| < K$ for all z then f(z) is constant.

PROOF: Let a & b be arbitrary distinct points in z-plane and let C be a large circle with center z = 0 and radius R such that C encloses a & b. The equation of C is

$|z| = R$ so that $z = Re^{i\theta}$
$dz = iRe^{i\theta}$
$|dz| = R\,d\theta$

f(z) is bounded for all z $\Rightarrow |f(z)| \leq m$ for all z where $m > 0$
By Cauchy integral formula

$$f(a) = \frac{1}{2\pi i}\int_c \frac{f(z)}{z-a}\,d(z)$$

$$f(b) = \frac{1}{2\pi i}\int_c \frac{f(z)}{z-b}\,d(z)$$

$$f(a) - f(b) = \frac{1}{2\pi i}\int_c \left(\frac{1}{z-a} - \frac{1}{z-b}\right) f(Z)\,d(Z)$$

$$f(a) - f(b) = \frac{a-b}{2\pi}\int_c \frac{f(z)\,d(z)}{(z-a)(z-b)}$$

$$|f(a) - f(b)| \leq \frac{|a-b|}{2\pi}\int_c \frac{|f(z)||d(z)|}{(|z|-|a|)(|z|-|b|)}$$

$$|f(a) - f(b)| \leq \frac{M|a-b|.2\pi R}{2\pi(R-|a|)(R-|b|)}$$

$$|f(a) - f(b)| \leq \frac{MR|a-b|}{(R-|a|)(R-|b|)} \to 0 as R \to \infty$$

$\therefore$ f(a) – f(b) = 0

"or" f(a) = f(b)
Showing there by f(z) is constant

[$\int_c |dz|$ = circumference of circle, $C = 2\pi R$]

The liouville's Theorem is one of the most outstanding Theorems in Complex Analysis which has no counter part in Real Analysis. In fact the Theorem does not hold for real function.

2. Proof Of The Fundamental Theorem Of The Algebra

We shall prove it by contradiction suppose p(z) $\neq 0$ for any value of z. Then

$$f(z) = \frac{1}{p(z)} = \frac{1}{a_0 + a_1 z + \ldots + a_n z^n}$$

$$f(z) = \frac{1}{z^n\left(\frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \frac{a_2}{z^{n-2}} + \ldots + a_n\right)} \to 0 as z \to \infty$$

$\therefore$ for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$|f(z)| < \varepsilon$ when $|z| < \delta$

Since f(z) is continuous in the bounded closed domain $|z| \leq \delta$ therefore f(z) is bounded in the closed domain $|z| \leq \delta$, so there exists a positive number k such that

$|f(z)| < k$ for $|z| \leq \delta$

If M = Max(ε , k) , Then we have

$|f(z)| = \left|\frac{1}{p(z)}\right| < M$ for every z

Hence by Liouville's Theorem f(z) is constant. This gives a contradiction. Since p(z) is not constant for n =1, 2, 3, and $a_n \neq 0$ Thus p(z) must be zero for at least one variable of z.

i.e. p(z) = 0 must have at least one root say α_1 then we have p(α_1) = 0

Now p(z) = p(z) – p(α_1)

i.e. $P(z) = (a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n)$

$- (a_0 + a_1 \alpha_1 + a_2 \alpha_1^2 + \ldots + a_n \alpha_1^n)$

"or" $p(z) = a_1(z - \alpha_1) + a_2(z^2 - \alpha_1^2) + \ldots + a_n(z^n - \alpha_1^n)$

i.e. $P(z) = (z - \alpha_1)\, p_1(z)$

where $p_1(z)$ is a polynomial of degree n-1. Again $p_1(z) = 0$ must have at least one root say α_2 (α_2 may be equal to α_1) proceeding as above we have

$P(z) = (z - \alpha_1)(z - \alpha_2) p_2(z)$

where $p_2(z)$ is a polynomial of degree n – 2 continuing in this way we see that p(z) = 0 has exactly n roots.

Second proof : For the second proof of the theorem we must know the following theorem known as Rouche's Theorem

Statement: If f(z) and g(z) are analytic inside and on a simple closed curve C and $|g(z)| < |f(z)|$ on C,
Then f(z) and f(z) + g(z) both have the same number of zeros inside C

Proof : Suppose f(z) and g(z) are analytic inside and on a simple closed curve C and

$|g(z)| < |f(z)|$ on C

Firstly we shall prove that neither f(z) nor f(z) + g(z) has zero on C
If f(z) has a zero at z = a on C then f(a) = 0

But $|g(z)| < |f(z)|$ on C

which is absurd Again if f(z) +g(z) has a zero at z = a on C
Then f(a) + g(a) = 0 so that

$$f(a) = -g(a)$$

"or"

$$|g(a)| = |f(a)|$$

Again we get a contradiction , Thus neither f(z) nor f(z) + g(z) has a zero on C
Let N_1 and N_2 be number of the zeros of f and f + g respectively inside C.
we know that f and f + g both are analytic within and on C and have no poles inside C.
Therefore, by usual formula.

$$\frac{1}{2\pi i}\int_c \frac{f'}{f}dz = N - P$$

gives

$$\frac{1}{2\pi i}\int_c \frac{f'}{f}dz = N_1$$

and

$$\frac{1}{2\pi i}\int_c \frac{f'+g'}{f+g}dz = N_2$$

subtracting we get

$$\frac{1}{2\pi i}\int_c \left(\frac{f'+g'}{f+g} - \frac{f'}{f}\right)dz = N_2 - N_1 \longrightarrow \text{(I)}$$

Taking $\frac{g}{f} = \phi$ so that $g = \phi f$

$$|g| < |f| \Longrightarrow \left|\frac{g}{f}\right| < 1 \Longrightarrow |\phi| < 1$$

Also $$\frac{f'+g'}{f+g}$$

$$= \frac{f' + f'\phi + \phi' f}{f + \phi f}$$

$$= \frac{f'(1+\phi) + \phi' f}{f(1+\phi)}$$

"or" $$\frac{f'+g'}{f+g} - \frac{f'}{f} = \frac{\phi'}{1+\phi}$$

Using in (I) we get

$$N_2 - N_1 = \frac{1}{2\pi i}\int_c \frac{\phi'}{1+\phi}dz$$

$$N_2 - N_1 = \frac{1}{2\pi i}\int_c \phi'(1+\phi)^{-1}dz \longrightarrow \text{(II)}$$

Since we have seen that $|\phi| < 1$ and so binomial expansion of $(1+\phi)^{-1}$ is possible and binomial expansion thus obtained is uniformly convergent and hence term by term integration is permissible, Hence

$$\int_c \phi'(1+\phi)^{-1}dz = \int_c \phi'[1-\phi+\phi^2-\phi^3+\ldots]dz$$

$$\int_c \phi'(1+\phi)^{-1}dz = \int_c \phi' dz - \int_c \phi'\phi dz + \int_c \phi'\phi^2 dz + \ldots \longrightarrow \text{(III)}$$

The function f and g both are analytic within and on C and f(z) ≠ 0 for any point on C . Hence $\frac{g}{f} = \phi$ is analytic and non–zero for any point on C. Therefore ϕ and it's all derivatives are analytic

∴ By caucly's integral theorem, each integral on R.H.S of (3) vanishes consequently.

$$\int_c \phi'(1+\phi)' dz = 0$$

In this event (2) takes the form

$N_2 - N_1 = 0$ or $N_1 - N_2$

3. Proof of Fundamental Theorem of Algebra

Consider the polynomial

$a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n$

such that $a_n \neq 0$

Take $f(z) = a_n z^n$

$g(z) = a_0 + a_1 z + \ldots + a_{n-1} z^{n-1}$

Let C be a circle $|z| = r$ where $r > 1$.

Then

$$|g(z)| \leq |a_0| + |a_1| r + |a_2| r^2 + \ldots + |a_{n-1}| r^{n-1}$$

$$|g(z)| \leq |a_0| r^{n-1} + |a_1| r^{n-1} + |a_2| r^{n-1} \ldots + |a_{n-1}| r^{n-1}$$

$$|g(z)| \leq [|a_0| + |a_1| + |a_2| + \ldots + |a_{n-1}|] r^{n-1}$$

But $|f(z)| = |a_n z^n| = |a_n| r^n$

$$\therefore \left|\frac{g(z)}{f(z)}\right| \leq \frac{[|a_0| + |a_1| + |a_2| + \ldots + |a_{n-1}|] r^{n-1}}{|a_n| r^n}$$

$$\left|\frac{g(z)}{f(z)}\right| = \frac{|a_0| + |a_1| + |a_2| + \ldots + |a_{n-1}|}{|a_n| r}$$

Now if |g(z)|<|f(z)| so that $\left|\frac{g(z)}{f(z)}\right| < 1$, then

$$\frac{|a_0|+|a_1|+|a_2|+\ldots+|a_{n-1}|}{|a_n|r} < 1$$

This

$$\Longrightarrow r > \frac{|a_0|+|a_1|+|a_2|+\ldots+|a_{n-1}|}{|a_n|}$$

Since r is arbitrary and hence by choosing r large enough, the last condition can be satisfied so that |g(z)|<|f(z)|. Now applying Rouche's theorem, we find that the given polynomial f(z) + g(z) has the same numbers of zeros as f(z) But $f(z) = a_n z^n$ has exactly n zeros all located at z = 0 . Consequently f(z) +g(z) has exactly n zeros. Consequently the given Polynomial has already n zeros.

Third Proof : For the proof we must know the following theorem known as Maximum Modulus principle**.**

Statement: Suppose f(z) is analytic within and on a simple closed countor C and f(z) is not constant . Then |f(z)| reaches its maximum value on C (and not inside C), that is to say , if M is the maximum value of |f(z)| on C, then |f(z)| < M for every z inside C.

PROOF: We prove this theorem by the method of contradiction, Analyicity of f(z) declares that f(z) is continuous within and on C. Consequently |f(z)| attains its maximum value M at the same point within or on C. we want to show that |f(z)| attains the value M at a point lying on the boundary of C (and not inside C) . Suppose, if possible, this value is not attained on the boundary of C but is attained at a point z = a within C so that
max|f(z)|=|f(a)| = M……………(1)

and |f(z)| ≤ M $\forall$ Z within C………..(2)
Describe a circle Γ with a as center lying within C. Now f(z) is not constant and its continuity implies the existence of a point z = b inside Γ such that |f(b)| < M

Let ε > 0 be such that |f (b)| = M - ε
Again |f(z)| is continous at z = b

and so |[|f(z)| - |f(b)|]| < $\frac{\varepsilon}{2}$

Whenever |z – b| < δ
Since
|[|f(z)| - |f(b)|]| ≥ |f(z)| - |f(b)|

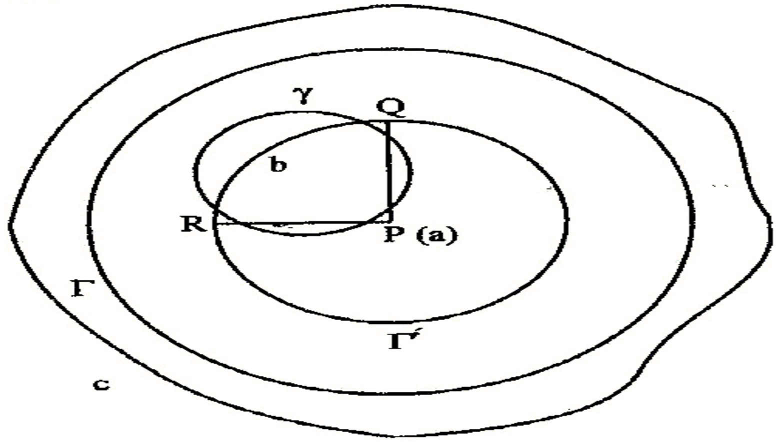
"or" |f(z)| - |f(b)| ≤ |[|f(z)| - |f(b)|]| < $\frac{\varepsilon}{2}$

"or" |f(z)| - |f(b)| < $\frac{\varepsilon}{2}$

"or" |f(z)| < |f(b)| + $\frac{\varepsilon}{2}$

= M - $\in$ + $\frac{\varepsilon}{2}$ = M - $\frac{\varepsilon}{2}$

"or" |f(z)| < M - $\frac{\varepsilon}{2}$ $\forall$ z s.t |z – b| < δ …(3)



We draw a circle γ with center at b and radius δ. Then (3) shows that

$|f(z)| < M - \dfrac{\varepsilon}{2}$ $\forall$ Z inside γ …………(4)

Again we draw another circle Γ́ with center at a and radius $|b - a| = r$

By Cauchy's Integral formula .

$$f(a) = \frac{1}{2\pi i}\int \frac{f(z)}{z-a}dz$$

on Γ́, $z - a = re^{i\theta}$

$$f(a) = \frac{1}{2\pi i}\int_0^{2\pi} f(a+re^{i\theta})\frac{rie^{i\theta}d\theta}{rie^{i\theta}}$$

If we measure θ in anti-clock wise direction & if

$< QPR = \alpha$ then

$$f(a) = \frac{1}{2\pi}\left[\int_0^{\alpha} + \int_{\alpha}^{2\pi}\right] f(a+re^{i\theta})d\theta$$

$$\therefore |f(a)| \le \frac{1}{2\pi}\int_0^{\alpha}\left|f(a)+re^{i\theta}\right|d\theta + \frac{1}{2\pi}\int_{\alpha}^{2\pi}\left|f(a+re^{i\theta})\right|d\theta$$

$$.|f(a)| < \frac{1}{2\pi}\int_0^{\alpha}\left(M - \frac{\in}{2}\right)d\theta + \frac{1}{2\pi}\int_{\alpha}^{2\pi} Md\theta$$

$$f(a) = \left(M - \frac{\in}{2}\right)\frac{\alpha}{2\pi} + M\frac{(2\pi - \alpha)}{2\pi} = M - \frac{\alpha \in}{4\pi}$$

Then $M = |f(a)| < M - \dfrac{\alpha \in}{4\pi}$

"or" $M < M - \dfrac{\alpha \in}{4\pi}$. A contradiction

For M cannot be less than $M - \dfrac{\alpha \in}{4\pi}$

Hence the Required results follow.

4. Minimum Modulus Principle

STATEMENT:- Suppose f(z) is analytic within and on a closed contour C and Let $f(z) \neq 0$ inside C suppose further that f(z) is not constant . Then $|f(z)|$ attains its minimum value at a point on the boundary of "C" that is to say, if M is the minimum value of $|f(z)|$ inside and on C. Then $|f(z)| > m \ \forall$ z inside C

PROOF: f(z) is analytic within and on C and $f(z) \neq 0$ inside C. It follows that $\frac{1}{f(z)}$ is analytic within C. By Maximum Modulus principal $\frac{1}{|f(z)|}$ attains its Maximum value on the boundary of C. So that $|f(z)|$ attains its Minimum value on the boundary of C. Hence the theorem

5. Proof Of The Fundamental Theorem Via Maximum

Modulus Principle Proof: Assume p(z) is non-constant and never zero. $\exists$ M such that $|p(z)| \geq |a_0| \neq 0$ if $|z| > M$. Since $|p(z)|$ is continuous, it achieves its minimum on a closed interval. Let z_0 be the value in the circle of radius M where p (z) takes its minimum value.

So $|p(z_0)| \geq |p(z)|$ for all $z \in C$, and in particular

$|p(z_0)| \geq |p(0)| = |a_0|$.

Translate the polynomial.

Let $p(z) = p((z - z_0) + z_0)$;

Let $p(z) = Q(z - z_0)$.

Note the minimum of Q occurs at

$z = 0$: $|Q(0)| \geq |Q(z)|$ for all $z \in C$.

$Q(z) = c_0 + c_j z^j + \cdots + c_n z^n$,

Where j is such that cj is the first coefficient (after c_0) that is Non-zero. I must show Q(0) = 0 Note if $c_0 = 0$, we are done.

We may rewrite such that

$$Q(z) = c_0 + c_j z^j + z^{j+1} R(z)$$

We will extract roots.

Let

$$re^{i\theta} = -\frac{c_0}{c_j}$$

Further, Let

$$z_1 = r^{\frac{1}{j}} + e^{\frac{i\theta}{j}}$$

So,
$$c_j z^j = -c_0$$

Let $\in > 0$ be a small real number. Then

$$Q(\in z_1) = c_0 + c_j \in^j z_1^{\ j} + \in^{j+1} z_1^{\ j+1} R(\in z_1)$$

$$|Q(\in z_1)| \leq |c_0 + c_j \in^j z_j^j| + \in^{j+1} |z_1|^{j+1} R(\in z_1)$$

$$|c_0| - \in^j |c_0| + \in^{j+1} |z_1|^{j+1} N,$$

Where N chosen such that $N > |R(\in z_1)|$, and $\in$ is chosen so that

$$\in^{j+1} |z_1|^{j+1} < \in^j |c_0|$$

Thus,

$$|Q(\in z_1)| < |c_0|,$$

But this was supposed to be our minimum. Thus , a contradiction. Hence proved

6. Proof of the Fundemental Theorem via Radius of convergence

We now prove the Fundamental theorem of Algebra: As always, p(z) is a non constant polynomial. Consider

$$f(z) = \frac{1}{p(z)} = b_0 + b_1 z + \ldots$$

and

$$p(z) = a_n z^n + \ldots + a_0, a_0 \neq 0$$

Lemma. $\exists c, r \in C$ such that $|b_k| > cr^k$ for infinitely many k.

Now, 1= p(z)f(z). Thus, $a_0 b_0 = 1$. This is our basic step .Assume we have some coefficient such that $|b_k| > cr^k$. We claim we can always find another .Suppose there are no more .Then the coefficient of z^{k+n} in p(z)f(z) is

$$a_0 b_{k+n} + a_1 b_{k+n-1} + \ldots + a_n b_k = 0$$

Thus, as we have $|b_j| > cr^j$ in this range ,we have the coefficient satisfies

$$|a_0| r^n + |a_1| r^{n+1} + \ldots + |a_{n-1}| r \leq |a_n|$$

f

$$r \leq \min\left\{1, \frac{|a_n|}{|a_0| + \ldots + |a_{n-1}|}\right\}$$

This will give that

$$|b_k| = \frac{|a_0 b_{k+n} + \ldots + a_{n-1} b_{k+1}|}{|a_n|}$$

$$|b_k| \leq \frac{|a_0 b_{k+n}| + \ldots + |a_{n-1} b_{k+1}|}{|a_n|} \leq cr^k$$

for sufficiently small.

Let z = $\frac{1}{r}$, Then

$$|b_k z^k| = \frac{|b_k|}{r^k} > c$$

This is true for infinitely many k, hence the power series diverges, contradicting the assumption that function is analytic and its power series converges everywhere.

7. Proof Of The Fundamental Theorem Via Picard's Theorem

Statement: If there are two distinct points that are not in the image of an entire function p(z) (ie. $\exists z_1 \neq z_2$ such that for all

$$z \in C, p(z) \neq z_1 or z_2),$$

then p(z) is constant.

We now prove the Fundamental Theorem of Algebra;

Let p(z) be a non-constant polynomial, and assume p(z) is never 0.

Claim:- If p(z) is as above , p(z) does not take on one of the variable $\frac{1}{k}$ for $k \in N$

Proof: Assume not. Thus, $\exists z_k \in C$ such that $p(z_k) = \frac{1}{k}$. If we take a circle D centered at the origin with sufficiently large radius, then $|p(z)| > 1$ for all z outside D. Thus each $z_1 \in D$, we have a convergent subsequence. Thus we have $z_{n_i} \to z'$ but

$$p(z') = \lim_{n_i \to \infty} p(z_{n_i}) = 0.$$

Thus there must be some k such that $p(z) \neq \frac{1}{k}$. Since there are two distinct values not in range of p, by picard's Theorem it is now constant. This contradicts our assumption that p(z) is non-constant. Therefore, $p(z_0) = 0$ for some z_0 .

Remark:- One can use a finite or countable versions of picards. Rather than missing just two points, we can modify the above to work if Picard instead stated that if we miss finitely many (or even countably) points, we are constant. Just look at the method above, gives $\frac{1}{k_1}$. We can then find another larger one, say $\frac{1}{k_2}$ .And so on. We can even get uncountably many such points by looking at numbers such as $\frac{\pi}{2}$ (using now the transcendence of C is 1).

8. Proof Of The Fundamental Theorem Via Cauchy's Integral Theorem

Statement:- Let f(z) be analytic inside and on the boundary of some region C. Then

$$\int_{\partial c} f(z) = 0$$

We now prove the Fundamental Theorem of Algebra.

Proof: Without loss of generality let p(z) be a non- constant polynomial and assume p(z) = 0. For $z \in R$ .assume $p(z) \in R$ (Otherwise, consider $p(z)\overline{p}(z)$.

Therefore, p(z) doesn't change signs for $z \in R$, or by the Intermediate Value Theorem it would have a zero.

$$\int_0^{2\pi} \frac{d\theta}{p(2\cos\theta)} \neq 0$$

This follows from our assumption that p(z) is of constant signs for real argument, bounded above from 0. This integral equals the contour integral

$$\frac{1}{i}\int_{|z|=1} \frac{dz}{zp(z+z^{-1})} = \frac{1}{i}\int_{|z|=1} \frac{z^{n-1}}{Q(z)}$$

If z $\neq 0$, Q(z) $\neq 0$

If z =0, then Q(z) $\neq 0$ Since

$P(z + z^{-1}) = a_n(z + z^{-1}) + \ldots$

$z^n p(z + z^{-1}) = z^n(\ldots a_n z^{-n}) + \ldots$

$= a_n + z(\ldots)$

Thus, $Q(z) = a_n$, which is non-zero . Hence, $Q(z) \neq 0$,

Consequently $\frac{z^{n-1}}{Q(z)}$ is analytic. By the Cauchy Integral

Formula $\frac{1}{i}\int_{|z|=1} \frac{z^{n-1}}{Q(z)} \neq 0.$ Thus, a contradiction!

9. The Fundamental Theorem Of Algebra

Our object is to prove that every non constant polynomial f(z) in one variable z over the complex numbers C has a root, i.e. that is a complex number r in C such that f(r) = 0. Suppose that

$f(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$

where n is at least 1, $a_n \neq 0$ and the coefficients a_i are fixed complex numbers. The idea of the proof is as follows: we first show that as |z| approaches infinity, |f(z)| approaches infinity as well. Since |f(z)| is a continuous function of z, it follows that it has an absolute minimum. We shall prove that this minimum must be zero, which establishes the theorem. Complex polynomial at a point where it does not vanish to decrease by moving along a line segment in a suitably chosen direction.
We first review some relevant facts from calculus.

10. Properties Of Real Numbers And Continuous Functions

Lemma 1. Every sequence of real numbers has a monotone (nondecreasing or nonincreasing) subsequence. Proof. If the sequence has some term which occurs infinitely many times this is clear. Otherwise, we may choose a subsequence in which all the terms are distinct and work with that. Hence, assume that all terms are distinct. Call an element "good" if it is bigger than all the terms that follow it. If there are infinitely many good terms we are done: they will form a decreasing subsequence. If there are only finitely many pick any term beyond the last of them. It is not good, so pick a term after it that is bigger. That is not good, so pick a term after it that is bigger. Continuing in this way (officially, by mathematical induction) we get a strictly increasing subsequence. That proves the theorem

lemma 2. A bounded monotone sequence of real numbers converges.

Proof. This is sometimes taken as an axiom about the reals. What is given here is an intuitive justification. We assume the sequence is non-decreasing: for the other case, take the negatives. The boundedness forces the integer parts of the terms in the sequence to stabilize, since a bounded montone sequence of integers is eventually constant. Thus, eventually, the terms have the form $m + f_n$ where m is a fixed integer, 0 is less than or equal to $f_n < 1$, and the f_n are non decreasing. The first digits of the f_n (after the decimal point) don't decrease and eventually stabilize: call the stable value a_1. For the f_n which begin with $.a_1$ ... the second digits increase and eventually stabilize: call the stable value a_2. Continuing in this fashion we construct a number $f = .a_1 a_2 \ldots a_h \ldots$ with the property that eventually all the f_n agree with it to as many decimal places as we like. It follows that f_n approaches f as n approaches infinity, and that the original sequence converges to m + f. That proves the theorem

Lemma 3. A bounded sequence of real numbers has a convergent subsequence. If the sequence is in the closed interval [a,b], so is the limit.

Proof. Using Fact 1, it has a monotone subsequence, and this is also bounded. By Fact 2, the monotone subsequence converges. It is easy to check that if all terms are at most b (respectively, at least a) then so is the limit. That proves the theorem

Lemma 4. A sequence of points inside a closed rectangle (say a is less than or equal to x is less than or equal to b, c is less than or equal to y is less than or equal to d) has a convergent subsequence. The limit is also in the rectangle.

Proof. Using Fact 3 we can pick out a subsequence such that the x-coordinates converge to apoint in [a,b]. Applying lemma 3 again, from this subsequence we can pick out a further subsequence such that the y-coordinates converge to a point in [c,d]. The x-coordinates of the points in this last subsequence still converge to a point in [a,b]. That proves the theorem

Lemma 5. A continuous real-valued function f defined on a closed rectangle in the plane is bounded and takes on an absolute minimum and an absolute maximum value.

Proof. We prove the result for the maximum: for the other case consider -f. For each integer n = 1, 2, 3, ... divide the rectangle into n^2 congruent sub rectangles by drawing n-1 equally spaced vertical lines and the same number of equally spaced horizontal lines. Choose a point in each of these sub rectangles (call these the special points at step n) and evaluate f at these points. From among these choose a point where f is biggest: call it P_n. The sequence $P_1, P_2, P_3, \ldots$ has a convergent subsequence: call it $Q_1, Q_2, Q_3, \ldots$, where

$Q_k = P_{\{n_k\}}$ Let Q be the limit of the sequence Q_k. It will suffice to show that f(Q) is bigger than or equal to f(P) for every point P in the rectangle. If not choose P in the rectangle such that f(P) > f(Q). For each k let P'_k be a special point at step n_k in a sub rectangle (among the $(n_k)^2$) that contains P. It follows that P'_k approaches P as k approaches infinity, since both sides of the sub rectangles approach zero as k approaches infinity. For every k, $f(Q_k)$ is at least $f(P'_k)$, by the choice of Q_k. Taking the limit as k approaches infinity gives a contradiction, since $f(Q_k)$ approaches f(Q) and, by the continuity of f, $f(P'_k)$ approaches f(P) as k approaches infinity. That proves the theorem. The result is valid for a continuous real- valued function on any closed bounded set in R^2 or R^n, where a set is closed if whenever a sequence of points in the set converges, the limit point is in the set.

Lemma 6. Let f be a continuous real- valued function on the plane such that f(x,y) approaches infinity as (x,y) approaches infinity. (This means that given any real number B, no matter how large, there is a real number m > 0 such that if $x^2 + y^2$ is at least m then f(x,y) is at least B.) Then f takes on an absolute minimum value at a point in the plane.}

Proof. Let B = f(0,0). Choose m > 0 such that if $x^2 + y^2$ is at least m then f(x,y) is at least B. Choose a rectangle that contains the circle of radius m centered at the origin. Pick Q in the rectangle so that the minimum value of f on the rectangle occurs at Q. Since (0,0) is in the rectangle f(Q) is at most B. Since outside the rectangle all values of f are at least B, the value of f at Q is a minimum for the whole plane, not just the rectangle. That proves the theorem

Lemma 7. Let g be a continuous function of one real variable which takes on the values c and d on a certain interval. Then g takes on every value r between c and d on that interval. Proof. Let g(a) = c and g(b) = d. We may assume without loss of generality that a < b. Replacing g by g - r we may assume that g is positive at one endpoint, negative at the other, and never vanishes. We shall get a contradiction. We shall construct a sequence of intervals $I_1 = [a,b]$, I_2, ..., I_n, ... such that $I_{[n+1]}$ is contained in I_n for each n, g has values of opposite sign at the end points of every I_n, and I_n has length $\frac{b-a}{2^{\{n-1\}}}$. In fact if I = $[a_n,b_n]$ has already been constructed and M is the midpoint of I_n, then g(M) has opposite sign from at least one of the numbers $g(a_n)$, $g(b_n)$, and so we can choose one of $[a_n, M]$ or $[M, b_n]$ for $I_{\{n+1\}}$. The a_n are non decreasing, the b_n are non increasing, and $a_n < b_n$ for all n. It follows that both sequences have limits. But $b_n - a_n$ approaches 0 as n approaches infinity, so the two limits are the same. Call the limit h. Since a_n approaches h, $g(a_n)$ approaches g(h). Similarly, $g(b_n)$ approaches g(h). Since $g(a_n)$ and $g(b_n)$ have opposite signs for all n, this can only happen if g(h) = 0. That proves the theorem

Remark:- Consider a polynomial f(x) with real coefficients of odd degree. Then lemma 7 implies that f has at least one real root. To see this, we may assume that f has a positive leading coefficient (otherwise, replace f by -f). It is then easy to see that f(x) approaches +infinity as x approaches + infinity while f(x) approaches -infinity as x approaches -infinity. Since f(x) takes on both positive and negative values, lemma 7 implies that f takes on the value zero.
We want to note that if u, u' are complex numbers then

$$|u + u'| \leq |u| + |u'|.$$

To see this note that, since both sides are non-negative, it suffices to prove this after squaring both sides, i.e. to show that $|u + u'|^2 \leq |u|^2 + 2|uu'| + |u'|^2$. Now, it is easy to see that for any complex number v,

$$|v|^2 = v(\overline{v}),$$

$\overline{v}$ where denotes the complex conjugate of v. Using this the inequality above is equivalent to

$$(u + u')\overline{(u + u')} \leq u\overline{u} + 2|uu'| + u'(\overline{u'}).$$

Multiplying out, and canceling the terms which occur on both sides, yields the equivalent inequality

$$u\overline{u'} + \overline{u}u' \leq 2|uu'| = 2|u||u'| = 2|u\overline{u}| = 2|u||\overline{u'}|$$

Let w = u(u')-. Then $\overline{w} = (\overline{u})(\overline{u'}) = (\overline{u})u'$.

Thus, what we want to show is that $w + \overline{w} \leq 2|\overline{w}|$.

If w = a + bi where a, b are real this becomes the assertion that $2a \leq 2\,\{a^2+b^2\}^{1/2}$

or $a \leq \sqrt{a^2 + b^2}$,

which is clear. Moreover, equality holds if and only if a is non negative and b is zero, i.e., if and only if $w = u(\overline{u'})$ is a non-negative real number.

We also get that $|u \pm u'| \geq |u| - |u'|$: replacing u' by if necessary we can assume the sign is -, and we already know that $|u| \leq |u-u'| + |u'|$, which is equivalent. Finally, we want to justify carefully why, when n is a positive integer, every complex number has an n^{th} root. First note that the fact holds for non-negative real numbers r using lamma 6 applied to the function F:R to R given by $F(x) = x^{n-r}$: the function is non-positive at x = 0 and positive for all sufficiently large x, and so takes on the value 0. We can now construct an n^{th} root for r($\cos t + i \sin t$) ,namely

$$r^{1/n} \cos\left(\frac{t}{n}\right) + i \sin\left(\frac{t}{n}\right),$$

Using De-Moivre's formula.

11. Proof of the fundamental theorem of algebra

Let $f(z) = a_n z^n + ... + a_0$, where the a^i are in C, n > 0, and a_n is not 0. If we let z = x + yi with x and y varying in R, we can think of f(z) as a C-valued function of x and y. In fact if we multiply out and collect terms we get a formula

$f(z) = P(x,y) + iQ(x,y)$

where P and Q are polynomials in two real variables x, y. We can therefore think of

$|f(x+yi)| = (P(x,y)^2 + Q(x,y)^2)^{1/2}$

as a continuous function of two real variables. We want to apply lamma 6 to conclude that it takes on an absolute minimum. Thus, we need to understand what happens as (x,y) approaches infinity. But we have

$$|f(z)| = |z^n| + |a_n| \left| 1 + b_{\frac{n-1}{z}} + b_{\frac{n-2}{z^2}} + \ldots + b_{\frac{0}{z^n}} \right|,$$

where $b_i = a_i/a_n$ for $0 \leq i \leq n-1$. Now

$$\left| 1 + \left(b_{\frac{\{n-1\}}{z}} + b_{\frac{\{n-2\}}{z^2}} + \ldots + b_{\frac{\{0\}}{z^n}} \right) \right| \quad \text{...(A)}$$
$$> |1| - \left| \left(b_{\frac{\{n-1\}}{z}} + b_{\frac{\{n-2\}}{z^2}} + b_{\frac{0}{z^n}} \right) \right|$$

The term that we are subtracting on the right is at most

$$\left| b_{\frac{\{n-1\}}{|z|}} \right| + \left| b_{\frac{\{n-2\}}{|z^2|}} \right| + \ldots + \left| b_{\frac{0}{|z^n|}} \right|,$$

and this approaches 0 at |z| approaches infinity. Hence, for all sufficiently large |z|, the quantity on the left in the inequality labeled (A) is at least 1/2, and so |f(z)| is at least $|z|^n \left| a_{\frac{n}{\frac{1}{2}}} \right|$ (1/2) for large |z|. Thus, |f(z)| approaches infinity as |z| approaches infinity. This implies, by lamma 6, that we can choose a point z = r = a + bi where |f(z)| is an absolute minimum. The rest of the argument is devoted to showing that f(r) must be zero. We assume otherwise and get a contradiction. To simplify calculations we are going to make several changes of variable. <u>Simplification 1.</u> Let g(z) = f(z+r), which is also a polynomial in z of degree n. g (resp. |g|) takes on the same set of values as f (resp. |f|). But |g| is minimum at z = 0, where the value is |f(0+r)| = |f(r)|. Thus, we may assume without loss of generality that |f| is minimum at z = 0. (We change back the notation and don't refer to g any more.) We are now assuming that $a_0 = f(0)$ is not 0. Let $a = |a_0|$. <u>Simplification 2</u>. Replace f by $(1/a_0)f$. All values of f are divided by a_0. All values of |f| are divided by a. The new function still has its minimum absolute value at z = 0. But now the minimum is 1. We still write f for the function. Thus, we can assume that f(0) is 1 (this means that $a_0 = 1$) and that 1 is the minimum of |f|. <u>Simplification 3</u>. We know that a_n is not 0. Let k be the least positive integer such that a_k is not 0. (It might be 1 or n.) We can write

$$f(z) = 1 + a_k z^k + \ldots + a_n z^n$$

We next observe that if we replace f(z) by f(cz) where c is a fixed nonzero complex number the set of values of f (and of |f|) does not change, nor does the constant term, and 0 = c(0) stays fixed. The new f has all the same properties as the old: its absolute value is still minimum at $z = 0$. It makes life simplest if we choose c to be a k^{th} root of $(-1/a_k)$. The new f we get when we substitute cz for z then turns out to be $1 - z^k + a'_{k+1} z^{k+1} + ... + a'_n z^n$. Thus, there is no loss of generality in assuming that $a_k = -1$. Therefore, we may assume that

$$f(z) = 1 - z^k + a_{k+1} z^{k+1} + ... + a_n z^n.$$

If $n = k$ then $f(z) = 1 - z^n$ and we are done, since $f(1) = 0$. Assume from here on that k is less than n. The main point. We are now ready to finish the proof. All we need to do is show with $f(z) = 1 - z^k + a_{k+1}z^{k+1} + ...+ a_nz^n$ that the minimum absolute value of f is less than 1, contradicting the situation we have managed to set up by assuming that f(r) is not 0. We shall show that the absolute value is indeed less than one when z is a small positive real number (as we move in the complex plane away from the origin along the positive real axis or x-axis, the absolute value of f(z) drops from 1.) To see this assume that z is a positive real number with $0 < z < 1$. Note that 1-z is then positive. We can then write

$$|f(z)| = |1 - z^k + a_{k+1}z^{k+1} + ... + a_nz^n|$$

$$\leq |1-z^k| + |a_{k+1}z^k + ... + a_nz^n|$$

$$= 1- z^k + | a_{k+1}z^{k+1} + ... + a_nz^n|$$

$$\leq 1- z^k + | a_{k+1}|z^{k+1} + . . . + |a_n|z^n$$

(keep in mind that z is a positive real number)

$$= 1 - z^k (1 - w_z), \text{ where } w_z = | a_{k+1}|z + ... + |a_n|z^{n-k}.$$

When z approaches 0 through positive values so does w_z. Hence, when z is a small positive real number, so is $z(1-w_z)$, and so for z a small positive real number we have that

$$0 < 1 - z^k(1-w_z) < 1.$$

Since $|f(z)| < 1 - z^k(1-w_z)$

it follows that $|f(z)|$ takes on values smaller than 1, and so $|f(z)|$ is not minimum at $z = 0$ after all. This is a contradiction. It follows that the minimum value of $|f(z)|$ must be 0, and so f has a root .That proves the theorem.

12. Fundemental Theorem of Algebra via Fermat's Last Theorem

For the proof of the theorem we must know the following lemmas

Lemma 1:If an algebraic equation f(x) has a root α, then f(x) can be divided by x-α without a remainder and the degree of the result f'(x) is less than the degree of f(x).

Proof:

Let $f(x) = x^n + a_1x^{n-1} + . . . + a_{n-1}x + a_n$

Let α be a root such that $f(\alpha) = 0$

Now, if we divide the polynomial by (x-α), we get the following

$f(x)/(x - \alpha) = f_1(x) + R/(x-\alpha)$

where R is a constant and $f_1(x)$ is a polynomial with order n-1.

Multiplying both sides with x-α gives us:

$f(x) = (x - \alpha)f_1(x) + R$

Now, if we substitute α for x we get:

$f(\alpha) = 0$ which means that the constant in the equation is 0 so $R = 0$. That proves the lemma.

Theorem: Fundamental Theorem of Algebra

For any polynomial equation of order n, there exist n roots r_i such that:

$xn + a_1 x^{n-1} + ... + a_{n-1} x + a_n = (x - r_1)(x - r_2) . . . (x - r_n)$

Proof: Let $f(x) = x_n + a_1 x^{n-1} + ... + a_{n-1}x + an$

We know that f(x) has at least one solution α_1.

Using Lemma 1 above, we know that:

$f(x)/(x - \alpha_1) = f'(x)$ where deg $f'(x) = n-1$.

So that we have:
$f(x) = (x - \alpha_1)f'(x)$
But we know $f'(x)$ has at least one solution α_2
$f'(x)/(x-\alpha_2) = f''(x)$ where deg $f''(x) = n-2$. And
$f(x) = (x - \alpha_1)(x - \alpha_2)f''(x)$
Eventually we get to the point where the degree of
fn(x)=1 .In this case, $f_n(x) = x - \alpha_n$. This establishes that there are n roots for a given equation f(x) where the degree is n. Putting this all together gives us:
$f(x) = (x - \alpha_1)(x - \alpha_2) \ldots (x - \alpha_n)$
Now, since f(x)=0 only when one of the values α_i=x, we see that the n roots α_i are the only solutions. So, we have proven that each equation is equal to n roots.One important point to remember is that the n roots are not necessarily distinct. That is, it is possible that $\alpha_i = \alpha_j$ where $i \neq j$. That proves the theorem

Fundamental theorem of Algebra due to Cauchy
We will prove
$f(z) = z^n + a_1 z^{n-1} + a_2 z^{n-1} + \ldots + a_n = 0$
where a_i are complex numbers $n \geq 1$ has a complex root.

<u>Proof:-</u> let $a_n \neq 0$ denote $z = x + iy$ x, y real Then the function

$$g(x,y) = |f(z)| = |f(x+iy)|$$

Is defined and continuous in R^2
Let $c = \sum_{j=1}^{n} |a_j|$ it is +ve using the triangle inequality
We make the estimation

$$|f(z)| = |z|^n \left| 1 + \frac{a_1}{z} + \frac{a2}{z^2} + \ldots + \frac{|a_n|}{z^n} \right|$$

$$|f(z)| \geq \left(1 - \frac{|a_1|}{|z|} - \frac{|a_2|}{|z|} + \ldots + \frac{|a_n|}{|z|^n} \right)$$

$$|f(z)| = |z|^n \left(1 - \frac{c}{|z|} \right) \geq \frac{1}{2}|z|^n$$

Being true for $|z| > \max(1, 2c)$ denote $r : \max\left(1, 2c, \sqrt[n]{2|a_n|}\right)$ conside the disk $x^2 + y^2 \leq r$ Because it is compact the function g(x,y) attains at a point $(x_0\, y_0)$ of the disk its absolute minimum value (infimum) in the disk if $|z| > r$ we have

$$g(x,y) = |f(z)| \geq \frac{1}{2}|z|^n > \frac{1}{2} r^n \geq \frac{1}{2}\left(\sqrt[n]{2|a_n|}\right) = |a_n| > 0$$

Thus

$$g(x,y) \leq g(0,0) = |a_n| < |f(z)| \; for |z| > r$$

Hence $g(x_0, y_0)$ is the absolute minimum of g(x,y) in the whole complex plane
we show that $g(x_0\, y_0) = 0$ therefore we make the antitheses that $g(x_0 y_0) > 0$

Denote $z_0 = x_0 + iy_0 = z_0 + UB$

$$f(z) = f(z_0 + u) = b_n + b_{n-1}U + \ldots + b_1 U^{n-1} + U^n$$

Then $b_n = f(z_0) \neq 0$ by the antithesis

More over denote

$$c_j = \frac{b_j}{b_n}(i = 12\ldots n), c_0 = \frac{1}{b_n}$$

And assume that $c_{n-1} = c_{n-2} = \ldots = c_{n-k+1} = 0$
But $c_{n-k} \neq 0$ then we may write

$$f(z) = b_n(1 + c_{n-k} + c_{n-k}u + c_{n-k-1}u^{k+1} + \ldots + c_0 u^k)$$

If $c_{n-k}u^k = p(\cos\alpha + i\sin\alpha)\ \&\ u = e^{(\cos\phi + \sin\phi)}$

Then

$$c_{n-k}u^k = pe^{n(\cos\alpha + k\phi) + i\sin(\alpha + k\phi)}$$

By Demorvie's identity choosing

$$\rho \leq 1\ \&\ \phi = \frac{\overline{\pi\alpha}}{k}$$ we get

$$c_{n-k}u^k = -pe^k$$ & can make the estimate

$$\left|c_{n-k-1}u^{k+1} + \ldots + c_0 u^n\right| \leq \left|c_{n-k-1}\right|e^{k+1} + \ldots + \left|c_0\right|c^n$$

$$\leq \left(\left|c_{n-k-1}\right| + \ldots + \left|c_0\right|e^{k+1} = \mathrm{Re}^{k+1}\right)$$

Where R is constant

Let now $\rho = \min\left(1, \sqrt[k]{\frac{i}{p}\cdot\frac{p}{2R}}\right)$ we obtain

$$|f(z)| = |b_n|\left|1 - pe^k + h(u)\right|$$

$$|f(z)| = |b_n||1 - pe^k + h(u)| +$$

$$f(z) \leq |b_n||1 - pe^k + \mathrm{Re}^{k+1}|$$

$$\leq b_n\left|1 - e^k\left(p - R\frac{p}{2R}\right)\right|$$

$$\leq \frac{|b_n|}{2} < |bn| = |f(z_0)|$$

Which results is impossible since $|f(z)|$ was absolute minimum. Consequently the antithesis is wrong & the proof is settled.

13. Another Proof Of Fundamental Theorem Of Algebra

The proof depends on the following four lemmas

Lemma1: Any odd-degree real polynomial must have a real root.
Proof:

We know from intermediate value theorem, suppose $p(x) \in R[x]$ with degree p(x) = 2k +1 and suppose the leading coefficient $a_n > 0$
(the proof is almost identical if $a_n > 0$). Then

$P(x) = a_n x^n$ + (lower terms)
And n is odd . Then,

(1) $\lim_{x\to\infty} p(x) = \lim_{x\to\infty} a_n x^n = \infty$ since $a_n > 0$.

(2) $\lim_{x\to-\infty} p(x) = \lim_{x\to-\infty} a_n x^n = -\infty$

since $a_n > 0$ and n is odd.
From (1) , p(x) gets arbitrarily large positively , so there exists an x_1 with $p(x_1) > 0$. Similarly, from (2) their exits an x_2 with $p(x_2) < 0$ A real polynomial is a continuous real – valued function for all $x \in R$, Since $p(x_1)\,p(x_2) < 0$, it follows by the intermediate value theorem that their exits an x_3 , between x_1 and x_2, such that $p(x_3) = 0$.

Lemma 2: Any degree two complex polynomials must have a complex root.

Proof: We know from consequence of the quadratic formula and of the fact that any complex number has a square root.
If $p(x) = ax^2 + bx + c,\ a \neq 0$, then the roots formally are

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a},\ x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

From DeMoivere's theorem every complex number has a square root, hence x_1 , x_2 exists in C. They of course may be same if $b^2 - 4ac = 0$.

Lemma 3: If every non-constant real polynomial has a complex root, then every non-constant complex polynomial has a complex root.
Proof: According to concept of the conjugate of complex polynomial
Let $P(x) \in R[x]$ and suppose that every non-constant real polynomial has at least one complex root. Let $H(x) = P(x)\overline{P}(x)$. from previous lemma $H(x) \in R[x]$. By supposition there exists a $z_0 \in C$ with $H(z_0) = 0$. then

$$P(z_0)\overline{P}(z_0) = 0,$$

And since C has no zero divisors, either

$$P(z_0) = 0 \text{ or } \overline{P}(z_0) = 0.$$

then previous lemmas

$$\overline{P}(z_0) = \overline{\overline{P}}(\overline{z_0}) = P(\overline{z_0}) = 0\,.$$

Therefore, $\overline{z_0}$ is root of p(x).

Lemma 4:-
Any non-constant real polynomial has a complex root.

Proof: Let $f(x) = a_0 + a_1 x + \ldots + a_n x^n \in R[x]$
with $n \geq 1$, $a_n \neq 0$. The proof is an induction on the degree n of f(x). Suppose $n = 2^m q$ where q is odd. We do the induction on m. if m = 0 then f(x) has odd degree and the theorem is true from lemma 1. Assume then that theorem is true for all degrees $d = 2^k q'$ where $k > m$ and q' is odd . Now assume that the degree of f(x) is $n = 2^m q$. Suppose F' is the splitting field for f(x) over R in which the roots are $\alpha_1,\ldots,\alpha_n$. We show that at least one of these roots must be in C. (In fact, all are in C but to prove the lemma we need only show at least one.)
Let $h \in Z$ and from the polynomial

$$H(x) = \prod_{i<j}(x - (\alpha_i + \alpha j + h\alpha_i\alpha_j))$$

This is in F'[x] we chose pairs of roots $\{\alpha_i, \alpha_j\}$, so the number of such pairs is the number of ways of choosing two elements out of $n = 2^m q$ elements. This is given by

$$\frac{(2^m q)(2^m q-1)}{2} = 2^m q(2^m q-1) = 2^{m-1} q'$$

With q'odd. Therefore, the degree of H(x) is $2^{m-1}q'$.

H(x) is a symmetric polynomial in the root $\alpha_1,\ldots,\alpha_n$. Since $\alpha_1,\ldots,\alpha_n$ are the roots of a real polynomial, from lemma 3 any polynomial in the splitting field symmetric field symmetric in these roots must be a real polynomial.
Therefore, $H(x) \in R[x]$ with degree $2^{m-1}q'$.By the inductive hypothesis, then, H(x) must have a complex root. This implies that there exists a pair $\{\alpha_i, \alpha_j\}$ with

$$\alpha_i + \alpha_j + h\alpha_i \alpha j \in C$$

Since h was an arbitrary integer, for any integer h_1 there must exists such a pair $\{\alpha_i, \alpha_j\}$ with

$$\alpha_i + \alpha_j + h\alpha_i \alpha j \in C$$

Now let h_1 vary over the integers. Since there are only finitely many such pairs $\{\alpha_i, \alpha_j\}$, it follows that there must be least two different integers h_1, h_2 such that

$$z_1 = \alpha_i + \alpha_j + h_1\alpha_i\alpha_j \in C$$

and

$$z_2 = \alpha_i + \alpha_j + h_2\alpha_i\alpha_j \in C$$

Then $$z_1 - z_2 = (h_1 - h_2)\alpha_i\alpha_j \in C$$

and since $h_1, h_2 \in Z \subset C$ it follows that $\alpha_i\alpha_j \in C$. But then $h_1\alpha_i\alpha_j \in C$, from which it follows that $\alpha_i\alpha_j \in C$.

Then,

$$P(x) = (x-\alpha_i)(x-\alpha_j) = x^2 - (\alpha_i + \alpha_j)x + \alpha_i\alpha_j \in C[x]$$

However, P(x) is then a degree–two complex polynomial and so from lemma 2 its roots are complex. Therefore, $\alpha_i\alpha_j \in C$ and therefore f(x) has a complex root.

It is now easy to give a proof of the Fundamental Theorem of Algebra. From lemma 4 every non constant real polynomial has a complex root. From lemma 3 if every non constant real polynomial has a complex root, then every non- constant complex polynomial has a complex root providing the Fundamental Theorem

REFERNCES

[1]. Open Mappings and the Fundamental Theorem of Algebra R. L. Thompson Mathematics Magazine, Vol. 43, No. 1.
[2]. (Jan., 1970), pp. 39-40.
[3]. What! Another Note Just on The Fundemental Theorem of Algebra ? R. M. Redheffer The American Mathematical
[4]. Monthly, Vol. 71, No. 2. (Feb., 1964), pp. 180-185
[5]. Another Proof of the Fundamental Theorem of Algebra Daniel J. Velleman Mathematics Magazine, Vol.70, No. 3. (Jun.,1997), pp.282-293.
[6]. [Euler and the Fundamental Theorem of Algebra William Dunham The College Mathematics Journal, Vol.22, No. 4 (Sep., 1991), pp.282-293.
[7]. [An Elementary Constructive Proof of the Fundamental Theorem of Algebra P.C. Rosenbloom The American Mathematical Monthly, Vol. 52, No. 10. (Dec., 1945), pp.562-570.
[8]. Proof of the Fundamental Theorem of Algebra J.L. Brenner; R.C. Lyndon The American Mathematical Monthly, Vol. 88, No.4. (Apr., 1981), pp.253-256.

[9]. [Yet Another Proof of the Fundamental Theorem of Algebra R.P. Boas,Jr. The American Mathematical Monthly, Vol. 71, No. 2. (Feb., 1964), p.180.
[10]. The Fundamental Theorem of Algebra Raymond Redheffer The American Mathematical Monthly, Vol. 64,
[11]. No. 8 (Oct.,1957), pp.582-585.
[12]. A Proof of the Fundamental Theorem of Algebra R.P.Boas, Jr. The American Mathematical Monthly,
[13]. Vol. 42, No. 8.(Oct.,1935), pp.501-502
[14]. An Easy Proof of the Fundamental Theorem of Algebra Charles Fefferman The American Mathematical
[15]. Monthly, Vol. 74, No. 7. (Aug.- Sep., 1967), pp.854-855[11] Eulers: The Masster of Us All William Dunham
[16]. The Mathematica Association of America Dolciani Mathematical Expositions No.22, 1999.
[17]. The Fundamental Theorem of Algebra Benjamin Fine Gerhard Rosenberger Springer New York
[18]. Mathematical Analysis S.C Malik, Savita Arora New Age International Publication
[19]. Functions of a Complex Variable Goyal & Gupta Pragati Prakajhan Publication
[20]. A.R. Schep, A Simple Complex Analysis and an Advanced Calculus Proof of the FundamentalTheorem
[21]. of Algebra, Am Math Monthly 116, Jan 2009,67-67
[22]. David Antin's translation of Heinrich Dorrie's 100 Great Problems of Elementary
[23]. A. R. Schep, A Simple Complex Analysis and an Advanced Calculus Proof of the Fundamental Theorem of
[24]. Algebra, Am Math Monthly 116, Jan 2009, 67-68.
[25]. Heinrich Dorrie, 100 Great Problems of Elementary Mathematics.
[26]. Complex Analysis Kunihiko Kodaira Cambridge Studies in advanced mathematics
[27]. Elementary Theory of Analytic Functions of one or Several Complex Variables Henri Cartan
[28]. Complex variables An introduction Carlos A. Berenstein Roger gar Springer-Verlag
[29]. Fundamental Theorems of Operator Algebra V.1 Elementary Theorem V.2 Advanced Theorem Kadison- Richard.
[30]. Fundamental Theorem of Algebra Fine and Gerhard
[31]. Fundamentals of College algebraic Miller
[32]. Fundamental concept of higher Algebra Albert.
[33]. Fundamental concept of Algebra Meserve.
[34]. Fundamental Concept of Algebra Chevalley. C
[35]. Fundemental Theorey of One Complex Variable Green and Krantz
[36]. Function of complex variable Franklin
[37]. Function of Complex Variable with application Phillips vig Theory & Technique carrier range.
[38]. Function of a Complex variable and some of their applications Fuchs and Shabat Smitnov & Lebeder
[39]. Function of complex variable Goursat E.
[40]. Function of complex variable Conway J.B
[41]. Function of one complex variable Macrobart TM
[42]. Foundation of the Theorem of Algebras Volume I & Volume II Hankoock. Harris
[43]. Complex Analysis Freitag Busam
[44]. Function of Several variables lang.