Matching Dominating Sets of Euler Totient Cayley Graphs

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Abstract

Graph Theory has been realized as one of the most useful branches of Mathematics of recent origin, finding widest applications in all most all branches of sciences, social sciences, computer science and engineering. Nathanson[3] paved the way for the emergence of a new class of graphs, namely,Arithmetic Graphs by introducing the concepts of Number Theory, particularly, the Theory of congruences in Graph Theory. Cayley graphs are another class of graphs associated with the elements of a group. If this group is associated with some arithmetic function then the Cayley graph becomes an arithmetic graph. The Cayley graph associated with Euler Totient function is called an Euler Totient Cayley graph and in this paper we study the matching domination parameters of Euler Totient Cayley graphs.

Keywords: Euler Totient Cayley Graph, Matching Dominating sets.

1. Introduction

Cayley graph

Let $(X,.)$ be a group and $S,$ a symmetric subset of X not containing the identity element e of $X.$ The graph G whose vertex set $V = X$ and edge set E = $\{ (g, gs) \ / \ s \in S\}$ is called the Cayley graph of X corresponding to the set S and it is denoted by $G(X, S).$ Madhavi [2] introduced the concept of Euler totient Cayley graphs and studied some of its properties.

Euler Totient Cayley Graph

For each positive integer n, let Z_n be the additive group of integers modulo n and S be the set of all numbers less than n and relatively prime to n. The Euler totient Cayley graph $G(Z_n, \varphi)$ is defined as the graph whose vertex set *V* is given by $Z_n = \{0,1,2, \dots\dots n-1\}$ and the edge set is given by $E = \{(x,y)/x - y \in S \ \text{ or } \ y - x \in S\}.$ Clearly as proved by Madhavi [2], the Euler totient Cayley graph $G(Z_n, \varphi)$ is

1. a connected, simple and undirected graph,
2. $\varphi(n)$ - regular and has $\frac{n.\varphi(n)}{2}$ edges,
3. Hamiltonian,
4. Eulerian for $n \geq 3,$
5. bipartite if n is even and
6. complete graph if n is a prime.

2. Matching Dominating Sets of Euler Totient Cayley Graphs

The theory of domination in Graphs introduced by Ore [4] and Berge [1] is an emerging area of research today. The domination parameters of Euler Totient Cayley graphs are studied by Uma Maheswari [6] and we present some of the results without proofs and can be found in [5].

Theorem 2.1: If n is a prime, then the domination number of $G(Z_n, \varphi)$ is 1.

Theorem 2.2: If n is power of a prime, then the domination number of $G(Z_n, \varphi)$ is 2.

Theorem 2.3: The domination number of $G(Z_n, \varphi)$ is 2, if $n = 2p$ where p is an odd prime.

Theorem 2.4: Suppose n is neither a prime nor $2p.$ Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots\dots . . p_k^{\alpha_k}$, where $p_1, p_2, \dots p_k$ are primes and $\alpha_1, \alpha_2, \dots . \alpha_k$ are integers ≥ 1. Then the domination number of $G(Z_n, \varphi)$ is given by $\gamma\big(G(Z_n, \varphi)\big) = \lambda + 1,$ where λ is the length of the longest stretch of consecutive integers in V, each of which shares a prime factor with $n.$ A matching in a graph $G(V, E)$ is a subset M of edges of E such that no two edges in M are adjacent. A matching M in G is called a perfect matching if every vertex of G is incident with some edge in M. Let G (V, E) be a graph. A subset D of V is said to be a dominating set of G if every vertex in $V-D$ is adjacent to a vertex in D. The minimum cardinality of a dominating set is called the domination number of G and is denoted by $\gamma(G)$. A dominating set D of G is said to be a matching dominating set if the induced subgraph $\langle D\rangle$ admits a perfect matching. The cardinality of the smallest matching dominating set is called the matching domination number and is denoted by γ_m.

Theorem 1: The matching domination number of $G(Z_n,\varphi)$ is 2, if n is a prime.

Proof: Let n be a prime. Then $G(Z_n,\varphi)$ is a complete graph. It is clear that $\{0\}$ is a minimal dominating set as it dominates all other vertices of $\{1,2,\dots\dots,n-1\}$. Therefore $\gamma\left(G(Z_n,\varphi)\right)=1$. For any $t\in\{1,2,\dots\dots,n-1\}$, vertex 0 is adjacent to vertex t. So if $D_m=\{0,t\}$, then the induced subgraph $\langle D_m\rangle$ admits a perfect matching with minimum cardinality. Hence D_m is a minimal matching dominating set of $G(Z_n,\varphi)$.Therefore $\gamma_m\left(G(Z_n,\varphi)\right)=2$. ■

Theorem 2: If n is power of a prime, then the matching domination number of $G(Z_n,\varphi)$ is 2.

Proof: Consider $G(Z_n,\varphi)$ for $n=p^\alpha$ where p is a prime. Then the vertex set V of $G(Z_n,\varphi)$ is given by $V=\{0,1,2,\dots.,p^\alpha-1\}$. This set V falls into disjoint subsets as below.

1. The set S of integers relatively prime to n,
2. The set M of multiples of p,
3. Singleton set$\{0\}$.Let $D_m=\{0,t\,/\,GCD\,(t,n)=1\}$ where $t\in\{0,1,2,\dots.,p^\alpha-1\}$. Then D_m becomes a minimum dominating set of $G(Z_n,\varphi)$ as in Theorem 2.2. Since GCD $(t,n)=1,\ t\in S,$ the vertices 0 and t are adjacent. This gives that $\langle D_m\rangle$ admits a perfect matching. So D_m is a matching dominating set of $G(Z_n,\varphi)$ of minimum cardinality. Hence it follows that $\gamma_m\left(G(Z_n,\varphi)\right)=2$. ■

Theorem 3: The matching domination number of $G(Z_n,\varphi)$ is 4 if $n=2p$, where p is an odd prime.

Proof: Let us consider the Euler totient Cayley graph $G(Z_n,\varphi)$ for $n=2p$, p is an odd prime. Then the vertex set $V=\{0,1,2,\dots..,2p-1\}$ falls into the following disjoint subsets.

1. The set S of odd numbers which are less than n and relatively prime to n,
2. The set M of non - zero even numbers,
3. The set D_m of numbers 0 and p.

Then $D_m=\{0,p\}$ becomes a minimum dominating set of $G(Z_n,\varphi)$ as in Theorem 2.3. Further the vertices in D_m are non-adjacent because GCD $(p,n)\neq 1$. This gives that $\langle D_m\rangle$ does not admit a perfect matching. So D_m is not a matching dominating set of $G(Z_n,\varphi)$. In order that D_m admits a perfect matching, we need to add at least two vertices adjacent to each of the vertex in D_m so that $\gamma_m\left(G(Z_n,\varphi)\right)\geq 4$.Let $D_m'=\{0,p,r,p+r\,/\,GCD\ (r,n)=1\}$. Then $D_m\subset D_m'$. This implies that D_m' is a dominating set of $G(Z_n,\varphi)$. Moreover, since GCD$(r,n)=1$, vertex 0 is adjacent to vertex r and vertex p is adjacent to vertex $p+r$. Further we have GCD $(p,n)\neq 1$. Hence 0 and p are non-adjacent and similarly r and $p+r$ are non-adjacent. So $\langle D_m'\rangle$ admits a perfect matching. Hence D_m' is a matching dominating set of minimum cardinality.Thus $\gamma_m\left(G(Z_n,\varphi)\right)=4$. ■

Theorem 4: Let n be neither a prime nor $2p$ and $n=p_1^{\alpha_1}p_2^{\alpha_2}\dots\dots.p_k^{\alpha_k}$, where $p_1,p_2,\ \dots p_k$ are primes and $\alpha_1,\alpha_2,\dots.\alpha_k$ are integers ≥ 1. Then the matching domination number of $G(Z_n,\varphi)$ is given by

$$\gamma_m\left(G(Z_n,\varphi)\right)=\begin{cases}\lambda+1 & \text{if } \lambda \text{ is odd}\\ \lambda+2 & \text{if } \lambda \text{ is even}\end{cases}$$

Where λ is the length of the longest stretch of consecutive integers in V each of which shares a prime factor with n.

Proof: Let us consider $G(Z_n,\varphi)$ for $n=p_1^{\alpha_1}p_2^{\alpha_2}\dots\dots.p_k^{\alpha_k}$ where n is neither a prime nor $2p$. The vertex set V of $G(Z_n,\varphi)$ is given by $V=\{0,1,2,\dots..,n-1\}$. Then the set V falls into disjoint subsets as follows.

1. The set S of integers relatively prime to n,
2. The set $X=\{S_i\}$, where S_i is a collection of consecutive positive integers such that for every x in S_i, GCD $(x,n)>1$,
3. The singleton set$\{0\}$.

Let S_λ be the largest set in X with maximum cardinality λ. Suppose

$S_\lambda = \{ x_1, x_2, \dots\dots, x_\lambda \}$, where GCD$(x_i, n) > 1$ for $i = 1, 2, 3, \dots, \lambda$. Then $D_m = \{0, 1, 2, \dots\dots, \lambda\}$ is a dominating set of minimum cardinality, as in Theorem 2.4. Now two cases arise.

Case 1: Suppose λ is an odd number.

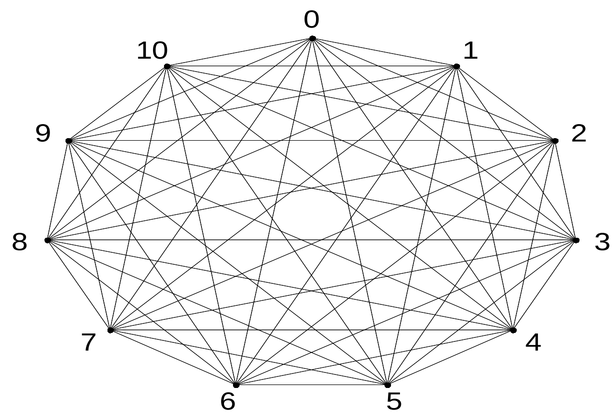
Consider the set $E_1 = \{(0,1), (2,3), \dots . (\lambda - 1, \lambda)\}$. Each pair $(2i, 2i + 1), 0 \leq i \leq \frac{\lambda-1}{2}$ is an edge of $G(Z_n, \varphi)$ as $(2i + 1) - 2i = 1 \in S$. So E_1 is a set of edges in $G(Z_n, \varphi)$. Obviously, no two edges in E_1 are adjacent. So $\langle E_1 \rangle$ admits a perfect matching. Hence D_m becomes a matching dominating set of $G(Z_n, \varphi)$. Since $\gamma(G(Z_n, \varphi)) = \lambda + 1$, we have $\gamma_m(G(Z_n, \varphi)) \geq \lambda + 1$. As $|D_m| = \lambda + 1$, it follows that D_m is a minimum matching dominating set of $G(Z_n, \varphi)$.

Therefore $\gamma_m(G(Z_n, \varphi)) = \lambda + 1$, if λ is an odd number.

Case 2: Suppose λ is an even number.

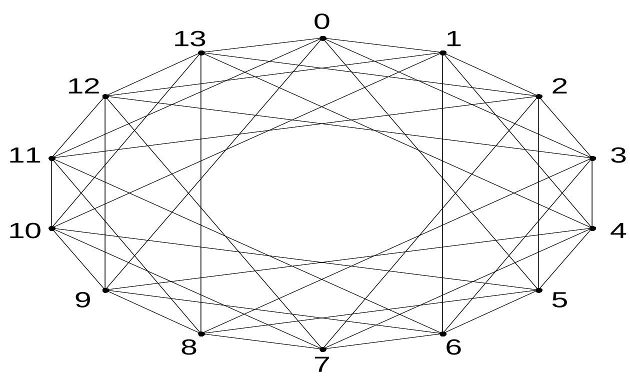
Let $D_m' = \{0, 1, 2, \dots\dots, \lambda, \lambda + 1\}$. Since $D_m \subset D_m'$ it follows that D_m' is a dominating set of $G(Z_n, \varphi)$. Consider the set $E_2 = \{(0,1), (2,3), \dots . (\lambda, \lambda + 1)\}$. As $(2i + 1) - 2i = 1 \in S$, each pair $(2i, 2i + 1), 0 \leq i \leq \frac{\lambda}{2}$ is an edge of $G(Z_n, \varphi)$. So E_2 is a set of edges in $G(Z_n, \varphi)$. Again it can be seen that no two edges in E_2 are adjacent. So $\langle E_2 \rangle$ admits a perfect matching. Hence D_m' is a matching dominating set of $G(Z_n, \varphi)$. Now λ is an even number implies that $\lambda + 1$ is an odd number. Since the matching domination number is always even it follows that $\gamma_m(G(Z_n, \varphi)) \geq \lambda + 2$. Therefore D_m' is a minimal matching dominating set of $G(Z_n, \varphi)$. Hence $\gamma_m(G(Z_n, \varphi)) = \lambda + 2$, if λ is an even number. ■

3. Illustrations



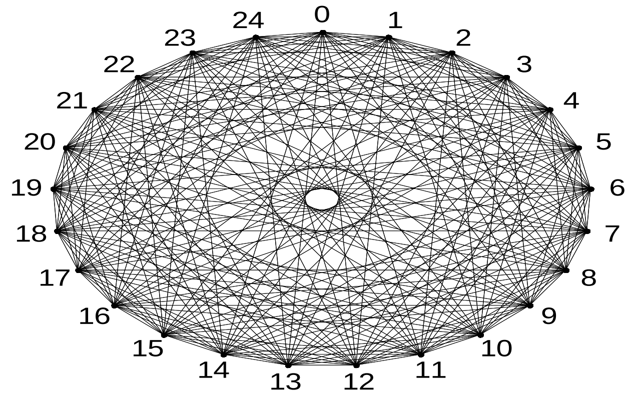
$G(Z_{11}, \varphi)$

Matching Dominating Set = {0, 1}



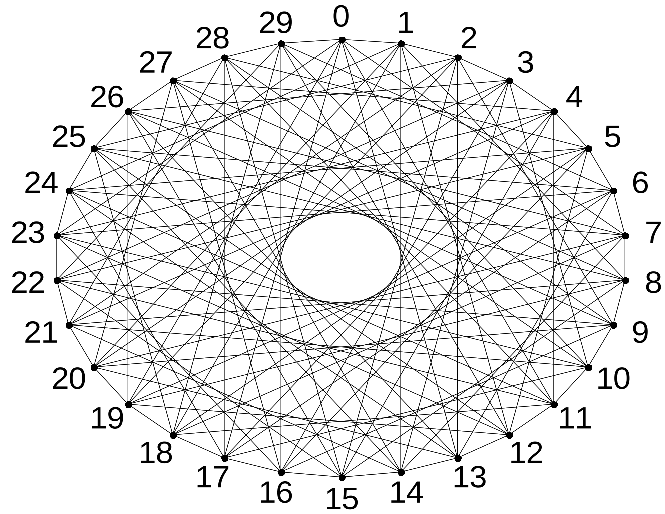
$G(Z_{14}, \varphi)$

Matching Dominating Set = {0, 1, 7, 8}



$G(Z_{25}, \varphi)$

Matching Dominating Set = {0, 1}



$G(Z_{30}, \varphi)$

Matching Dominating Set = {0, 1, 2, 3, 4, 5}

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