# Location Of The Zeros Of Polynomials

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**Abstract:** In this paper we prove some results on the location of zeros of a certain class of polynomials which among other things generalize some known results in the theory of the distribution of zeros of polynomials.

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## 1. Introduction And Statement Of Results

A celebrated result on the bounds for the zeros of a polynomial with real coefficients is the following theorem ,known as Enestrom –Kakeya Thyeorem[1,p.106]

**Theorem A:** If $0 < a_0 \le a_1 \le ...... \le a_n$, then all the zeros of the polynomial

$$P(z) = a_0 + a_1 z + a_2 z^2 + ...... + a_{n-1} z^{n-1} + a_n z^n$$

lie in $|z| \le 1$ .

Regarding the bounds for the zeros of a polynomial with leading coefficient unity, Montel and Marty [1,p.107] proved the following theorem:

**Theorem B:** All the zeros of the polynomial

$$P(z) = a_0 + a_1 z + a_2 z^2 + ...... + a_{n-1} z^{n-1} + z^n$$

lie in $|z| \le \max(L, L^{\frac{1}{n}})$ where L is the length of the polygonal line joining in succession the points

$0, a_0, a_1, ......, a_{n-1,1}; i.e.$

$$L = |a_0| + |a_1 - a_0| + ......+ |a_{n-1} - a_{n-2}| + |1 - a_{n-1}| .$$

Q .G. Mohammad [2] proved the following generalization of Theorem B:

**Theorem C:** All the zeros of the polynomial 0f Theorem A lie in

$$|z| \le R = \max( L_p, L_p^{\frac{1}{n}})$$

where

$$L_p = n^{\frac{1}{q}} (\sum_{j=0}^{n-1} |a_j|^p)^{\frac{1}{p}}, p^{-1} + q^{-1} = 1 .$$

The bound in Theorem C is sharp and the limit is attained by

$$P(z) = z^n - \frac{1}{n}(z^{n-1} + z^{n-2} + ......+ z + 1) .$$

Letting $q \to \infty$ in Theorem C, we get the following result:

**Theorem D:** All the zeros of P(z) 0f Theorem A lie in $|z| \le \max( L_1, L_1^{\frac{1}{n}})$ where

$$L_1 = \sum_{i=0}^{n-1} |a_i|) \ .$$

Applying Theorem D to the polynomial (1-z)P(z) , we get Theorem B.
Q .G. Mohammad , in the same paper , applied Theorem D to prove the following result:

**Theorem E:** If $0 < a_{j-1} \le k a_j, k > 0$, then all the zeros of

$$P(z) = a_0 + a_1 z + a_2 z^2 + \ldots\ldots + a_{n-1} z^{n-1} + a_n z^n$$

lie in $|z| \le \max(M, M^{\frac{1}{n}})$ where

$$M = \frac{(a_0 + a_1 + \ldots\ldots + a_{n-1})}{a_n}(k-1) + k\,.$$

The aim of this paper is to give generalizations of Theorems C and E. In fact, we are going to prove the following results:

**Theorem 1:** All the zeros of the polynomial

$$P(z) = a_0 + a_1 z + a_2 z^2 + \ldots\ldots + a_\mu z^\mu + z^n, 0 \le \mu \le n-1$$

lie in

$$|z| \le R = \max(L_p, L_p^{\frac{1}{n}})$$

where

$$L_p = n^{\frac{1}{q}} (\sum_{j=0}^{\mu} |a_j|^p)^{\frac{1}{p}}, p^{-1} + q^{-1} = 1\,.$$

**Remark 1:** Taking $\mu$ =n-1, Theorem 1 reduces to Theorem C.

**Theorem 2:** If $0 < a_{j-1} \le k a_j, k > 0$, then all the zeros of

$$P(z) = a_0 + a_1 z + a_2 z^2 + \ldots\ldots + a_\mu z^\mu + a_n z^n, 0 \le \mu \le n-1,$$

lie in $|z| \le \max(M, M^{\frac{1}{n}})$ where

$$M = \frac{(a_0 + a_1 + \ldots\ldots + a_\mu)}{a_n}(k-1) + k\,.$$

**Remark 2:** Taking $\mu$ =n-1 , Theorem 2 reduces to Theorem E and taking $\mu$ =n-1 , k=1, Theorem 2 reduces to Theorem A due to Enestrom and Kakeya..

## 2. Proofs Of Theorems

**Proof of Theorem 1.** Applying Holder's inequality, we have

$$|P(z)| = |a_0 + a_1 z + a_2 z^2 + \ldots\ldots + a_\mu z^\mu + z^n|$$

$$\ge |z|^n \left[ 1 - \sum_{j=1}^{\mu+1} |a_{j-1}| \frac{1}{|z|^{n-j+1}} \right]$$

$$\ge |z|^n \left[ 1 - n^{\frac{1}{q}} (\sum_{j=1}^{\mu+1} |a_{j-1}|^p \frac{1}{|z|^{(n-j+1)p}})^{\frac{1}{p}} \right].$$

If $L_p \ge 1, \max(L_p, L_p^{\frac{1}{n}}) = L_p$. Let $|z| \ge 1$. Then $\frac{1}{|z|^{(n-j+1)p}} \le \frac{1}{|z|^p}, j = 1,2,\ldots\ldots,\mu+1$.

Hence it follows that for $|z| > L_p$,

$$|P(z)| \geq |z|^n \left[ 1 - \frac{n^{\frac{1}{q}}}{|z|} \left( \sum_{j=0}^{\mu} |a_j|^p \right)^{\frac{1}{p}} \right] = |z|^n \left[ 1 - \frac{L_p}{|z|} \right] > 0.$$

Again if $L_p \leq 1, \max(L_p, L_p^{\frac{1}{n}}) = L_p^{\frac{1}{n}}$. Let $|z| \leq 1$. Then

$$\frac{1}{|z|^{(n-j+1)p}} \leq \frac{1}{|z|^{np}}, \; j = 1, 2, \ldots\ldots, \mu + 1.$$

Hence it follows that for $|z| > L_p^{\frac{1}{n}}$,

$$|P(z)| \geq |z|^n \left[ 1 - \frac{n^{\frac{1}{q}}}{|z|^n} \left( \sum_{j=0}^{\mu} |a_j|^p \right)^{\frac{1}{p}} \right] = |z|^n \left[ 1 - \frac{L_p}{|z|^n} \right] > 0.$$

Thus P(z) does not vanish for $|z| > \max(L_p, L_p^{\frac{1}{n}})$ and hence the theorem follows.

**Proof of Theorem 2.** Consider the polynomial

$$F(z) = (k - z)P(z) = (k - z)(a_0 + a_1 z + \ldots\ldots + a_\mu z^\mu + a_n z^n)$$
$$= ka_0 + (ka_1 - a_0)z + (ka_2 - a_1)z^2 + \ldots\ldots + (ka_\mu - a_{\mu-1})z^\mu - a_\mu z^{\mu+1}$$
$$+ ka_n z^n - a_n z^{n+1}$$

Applying Theorem C to the polynomial $\frac{F(z)}{a_n}$, we find that

$$L_1 = \frac{k(a_0 + a_1 + \ldots\ldots + a_\mu) - (a_0 + a_1 + \ldots\ldots + a_{\mu-1} + a_\mu) + ka_n}{a_n}$$

$$= \frac{(k-1)(a_0 + a_1 + \ldots\ldots + a_\mu)}{a_n} + k$$

=M

and the theorem follows.

## References


**[1]** M. Marden , The Geometry of Zeros,, Amer.Math.Soc.Math.Surveys ,No.3 ,New York 1949.

[2] Q.G. Mohammad , Location of the Zeros of Polynomials , Amer. Math. Monthly , vol.74,No.3, March 1967.