# Convexity of Minimal Total Dominating Functions Of Quadratic Residue Cayley Graphs

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## Abstract

Nathanson [17] paved the way for the emergence of a new class of graphs, namely,Arithmetic Graphs by introducing the concepts of Number Theory, particularly, the Theory of congruences in Graph Theory. Cayley graphs are another class of graphs associated with the elements of a group. If this group is associated with some arithmetic function then the Cayley graph becomes an arithmetic graph. Domination theory is an important branch of Graph Theory and has many applications in Engineering, Communication Networks and many others. In this paper we study the minimal total dominating functions of Quadratic Residue Cayley graphs and discuss the convexity of these functions in different cases.

**Keywords:** Arithmetic graph, Cayley graph, Total dominating set, Neighbourhood set, Quadratic Residue Cayley Graph, Total Dominating Functions, Minimal Total Dominating Functions, Convexity.

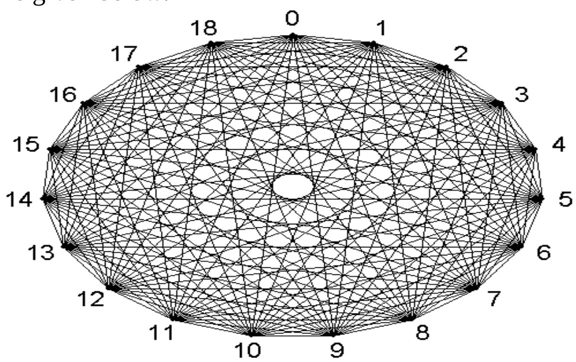
## 1. Introduction

There is a class of graphs, namely, Cayley graphs, whose vertex set V is the set of elements of a group (G, .) and two vertices x and y of G are adjacent if and only if $xy^{-1}$ is in some symmetric subset S of G. A subset S of a group (G, .) is called a symmetric subset of G if $s^{-1}$ is in S for all s in S. If the group (G, .) is the additive group ($Z_n$ , $\oplus$) of integers 0,1,2,……..,n-1 modulo n, and the symmetric set S is associated with some arithmetic function, then the Cayley Graph may be treated as an arithmetic graph. In this paper we consider Quadratic Residue Cayley graphs. A detailed study of convexity and minimality of dominating functions and total dominating functions are given in Cockayne et al. [2,3-12] Chesten et al. [1], Yu [18] and Domke et al. [13,14]. Algorithmic complexity results for these parameters are given in Laskar et al. [15] and Cockayne et al.[3].We start with the definition of a Quadratic Residue Cayley graph.

### Quadratic Residue Cayley Graph

Let p be an odd prime and n, a positive integer such that $n \not\equiv 0 \pmod p$. If the quadratic congruence, $x^2 \equiv n \pmod p$ has a solution then, n is called a quadratic residue mod p.

The Quadratic Residue Cayley graph G($Z_p$ , Q), is the Cayley graph associated with the set of quadratic residues modulo an odd prime p, which is defined as follows. Let p be an odd prime, S, the set of quadratic residues modulo p and let S* = {s, n - s / s$\in$ S, s $\neq$ n }. The quadratic residue Cayley graph G($Z_p$ , Q) is defined as the graph whose vertex set is $Z_p$ = { 0 , 1 , 2 , …… p – 1} and the edge set is E = { ( x , y ) / x – y or y - x $\in$ S*}.
For example the graph of G($Z_{19}$, Q) is given below.



## 2. Total Dominating Functions

**Total Dominating Set :** Let G(V, E) be a graph without isolated vertices. A subset T of V is called a total dominating set (TDS) if every vertex in V is adjacent to at least one vertex in T.

**Minimal Total Dominating Set :** If no proper subset of T is a total dominating set, then T is called a minimal total dominating set (MTDS) of G.

**Neighbourhood Set :** The open neighbourhood of a vertex u is the set of vertices adjacent to u and is denoted by N(u).

**Total Dominating Function :** Let G(V, E) be a graph without isolated vertices. A function $f : V \to [0,1]$ is called a total dominating function (TDF) if $f(N(v)) = \sum_{u \in N(v)} f(u) \geq 1$ for all $v \in V$.

**Minimal Total Dominating Function :** Let *f* and *g* be functions from *V*→ [0,1]. We define $f < g$ if $f(u) \leq g(u)$, $\forall$ $u \in V$, with strict inequality for at least one vertex u. A TDF f is called a minimal total dominating function (MTDF) if for all $g < f$, g is not a TDF.

We require the following results whose proofs are presented in [16].

**Lemma 1:** The Quadratic Residue Cayley graph $G(Z_p, Q)$ is $|S^*|$ - regular, and the number of edges in $G(Z_p, Q)$ is $\frac{|Z_p||S^*|}{2}$.

**Theorem 1:** The Quadratic Residue Cayley graph $G(Z_p, Q)$ is complete if p is of the form 4m+3.

Suppose p = 4m + 3. Then $G(Z_p, Q)$ is complete. Then each vertex is of degree p – 1. That is the graph $G(Z_p, Q)$ is (p – 1) – regular.

$\therefore |S^*| = p - 1$.

Hence each N(v) consists of p-1 vertices , $\forall v \in V$.

We consider the case p = 4m+3 of $G(Z_p, Q)$ and prove the following results.

## 3. MAIN RESULTS

**Theorem 3.1:** Let T be a MTDS of $G(Z_p, Q)$. Let $f : V \to [0,1]$ be a function defined by

$$f(v) = \begin{cases} 1, & if\ v \in T, \\ 0, & if\ v \in V - T. \end{cases}$$

Then f becomes a MTDF of $G(Z_p, Q)$.

**Proof:** Consider $G(Z_p, Q)$. Let T be a MTDS of $G(Z_p, Q)$.

Since $G(Z_p, Q)$ is complete, $|T| = 2$.

Also every neighbourhood N(v) of $v \in V$ consists of (p-1) –vertices.

Let f be a function defined on V as in the hypothesis.

Then the summation values taken over the neighbourhood N(v) of $v \in V$ is

$$\sum_{u \in N(v)} f(u) = \begin{cases} 2, & if\ u \in V - T, \\ 1, & if\ u \in T. \end{cases}$$

Therefore

$$\sum_{u \in N(v)} f(u) \geq 1, \forall\ v \in V.$$

This implies that f is a TDF.

We now check for the minimality of f.

Define a function $g : V \to [0,1]$ by

$$g(v)=\begin{cases} r, & if\ v\in T,\ v=v_k, \\ 1, & if\ v\in T-\{v_k\}, \\ 0, & if\ v\in V-T. \end{cases}$$

where $0 < r < 1$ and $v_k \in V$.
Since strict inequality holds at the vertex $v = v_k \in T$ of V, it follows that $g < f$.
Then

$$\sum_{u\in N(v)} g(u) = \begin{cases} 1+r, & if\ \ v\in V-T, \\ 1, & if\ \ v\in T,\ v=v_k, \\ r, & if\ \ v\in T,\ v\neq v_k. \end{cases}$$

Thus $\sum_{u\in N(v)} g(u) \not\geq 1,\ \ \forall\ v\in V.$

This implies that g is not a TDF. Since $r < 1$ is arbitrary it follows that there exists no $g < f$ such that g is a TDF.
Thus f is a MTDF.

**Theorem 3.2:** Let $T_1$ and $T_2$ be two MTDSs of $G(Z_p, Q)$ and $f_1,\ \ f_2$ be two functions of $G(Z_p, Q)$ defined by

$$f_1(v)=\begin{cases} 1,\ if\ v\in T_1, \\ 0,\ otherwise. \end{cases}$$

and $$f_2(v)=\begin{cases} 1,\ if\ v\in T_2, \\ 0,\ otherwise. \end{cases}$$

Then the convex combination of $f_1$ and $f_2$ becomes a TDF of $G(Z_p, Q)$ but not minimal.

**Proof:** Let $T_1$ and $T_2$ be two MTDSs of $G(Z_p, Q)$ and $f_1$ , $f_2$ be the functions defined as in the hypothesis. Then by Theorem 3.1, the above functions are MTDFs of $G(Z_p, Q)$.
Let $h(v)=\alpha f_1(v)+\beta f_2(v)$, where $\alpha+\beta=1\ and\ 0<\alpha<1,\ 0<\beta<1.$

**Case 1:** Suppose $T_1$ and $T_2$ are such that $T_1 \cap T_2 \neq \phi$.
Then the possible values of h(v) are

$$h(v)=\begin{cases} \alpha, & if\ v\in T_1\ and\ v\notin T_2, \\ \beta, & if\ v\in T_2\ and\ v\notin T_1, \\ \alpha+\beta, & if\ \ v\in \{T_1\cap T_2\}, \\ 0, & otherwise. \end{cases}$$

Since each neighbourhood N(v) of v in V consists of (p-1) vertices of $G(Z_p, Q)$, the summation value of h(v) taken over N(v) is

$$\sum_{u\in N(v)} h(u)=\begin{cases} \alpha+\beta+\beta, & if\ v\in T_1\ and\ v\notin T_2, \\ \alpha+\beta+\alpha, & if\ v\in T_2\ and\ v\notin T_1, \\ \alpha+\beta, & if\ \ v\in \{T_1\cap T_2\}, \\ 2(\alpha+\beta), & otherwise. \end{cases}$$

This implies that $\sum_{u\in N(v)} h(u)\geq 1$, $\forall$ $v \in V$.

Therefore h is a TDF. We now check for the minimality of h.
Define a function $g: V\to[0,1]$ by

$$g(v)=\begin{cases}\alpha, & if\ v\in T_1\ and\ v\notin T_2,\\ \beta, & if\ v\in T_2\ and\ v\notin T_1,\\ r, & if\ v\in \{T_1\cap T_2\},\\ 0, & otherwise.\end{cases}$$

where 0 < r < 1.

Since strict inequality holds at $v\in \{T_1\cap T_2\}$, it follows that g < h.

Now $$\sum_{u\in N(v)} g(u)=\begin{cases}r+\beta, & if\ v\in T_1\ and\ v\notin T_2,\\ \alpha+r, & if\ v\in T_2\ and\ v\notin T_1,\\ \alpha+\beta, & if\ v\in \{T_1\cap T_2\},\\ r+\alpha+\beta, & otherwise.\end{cases}$$

where $r+\beta<1+\beta$ and $\alpha+r< \alpha+1$.

Thus $\sum_{u\in N(v)} g(u)\geq 1, \quad \forall v\in V$.

Therefore g is a TDF. Hence it follows that h is a TDF but not minimal.

**Case 2:** Suppose $T_1$ and $T_2$ are disjoint.

Then the possible values of h(v) are

$$h(v)=\begin{cases}\alpha, & if\ v\in T_1,\\ \beta, & if\ v\in T_2,\\ 0, & otherwise.\end{cases}$$

Since each neighbourhood N(v) of v in V consists of (p-1) vertices of $G(Z_p, Q)$, the summation value of h(v) taken over N(v) is

$$\sum_{u\in N(v)} h(u)=\begin{cases}\alpha+2\beta, & if\ v\in T_1,\\ \beta+2\alpha, & if\ v\in T_2,\\ 2(\alpha+\beta), & otherwise.\end{cases}$$

This implies $\sum_{u\in N(v)} h(u)\geq 1$, $\forall$ v $\in$ V, since $\alpha+\beta=1$.

Therefore h is a TDF. We now check for the minimality of h.

Define a function $g:V\rightarrow[0,1]$ by

$$g(v)=\begin{cases}r, & if\ v\in T_1, v=v_i,\\ \alpha, & if\ v\in T_1, v\neq v_i\ ,\\ \beta, & if\ v\in T_2,\\ 0, & otherwise.\end{cases}$$

where 0 < r < $\alpha$.

Since strict inequality holds at $v=v_i\in T_1$, it follows that g < h.

Then $\sum_{u \in N(v)} g(u) = \begin{cases} \alpha + 2\beta, & if\ v \in \mathrm{T}_1, v = v_i, \\ r + 2\beta, & if\ v \in \mathrm{T}_1, v \neq v_i\ , \\ r + \alpha + \beta, & if\ \ v \in \mathrm{T}_2, \\ r + \alpha + 2\beta, & otherwise. \end{cases}$

where $r + 2\beta \leq \alpha + 2(1-\alpha) = 2 - \alpha > 1$.

i.e., $r + 2\beta > 1$.

Thus $\sum_{u \in N(v)} g(u) > 1, \quad \forall v \in V$.

Therefore g is a TDF. Hence it follows that h is not a MTDF.

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