Electromechanical Dynamics of simply-supported micro-plates

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Abstract:
This paper presents electromechanical coupled dynamic model of micro- plate subjected to electrostatic excitation. Equations of motion of laterally deformed thin plate are obtained analytically. The static displacement and dynamic characteristics of micro-plate are depicted as closed-form solutions and a reduced-order model based on Galerkin's approximate method is used for comparison of the result. A simply supported micro-plate model is considered to illustrate the methodology. The effect of applied voltage, the gap height and plate dimensions on the natural frequencies are illustrated.

Keywords: Electrostatic excitation, Rectangular plate, Simply-supported conditions, Analytical solution, Reduced-order model, Pull-in voltage, Natural frequency, Dynamic response.

1. Introduction

Micro Electro Mechanical System (MEMS) devices mostly employ movable parts such as thin beams, plates, rings, membranes and other mechanical components. Owing to their small size, significant forces and deformations can be obtained even with application of low voltages. Especially vibrations of micro-plates are used in devices like synthetic micro- jets, micro- speakers and gyroscopes of different structural types. Micro-plates are commonly actuated by piezoelectric or electrostatic sources for various resonance applications. Electrostatic actuation is most frequently applied principle and in order to control the performance, it is required to study the electromechanical dynamics of the plate. As the capacitive force changes with plate deflection, the electrical and mechanical forces are considered to be coupled. The applied electrostatic load has an upper limit beyond which the mechanical restoring force of micro-plate can no longer resist electrostatic load and it leads to collapse of the structure. Prediction of such a pull-in state is essential in design process to analyze sensitivity, frequency response and dynamic range of micro-sensors. In certain MEMS-based capacitive-type applications, like in micro-phones and pressure sensors, it is essential to avoid this pull-in effect since the contact between the two plates induces a short circuit and therefore renders the device inoperable. The literature contains many investigations [1-6] related to pull-in effect of micro-plates. When rate of voltage variation is considerable, the effect of inertia force has also to be considered and pull-in instability related to this situation (known as dynamic pull-in) is affected by different parameters such as electrostatic actuation, structural stiffness, inertia forces and damping. More emphasis is given recently to the numerical analysis of pull-in behaviour in micro-plates. Chao *et al*.[7] derived a closed form solution for prediction of pull-in voltage which offer some design guidelines for the device prior to its production. Zand and Ahmadian [8] presented a coupled finite element and finite difference model to analyze pull-in phenomenon, vibration behavior and dynamics of multi-layer micro-plates. First-order shear deformation theory has been employed to model dynamical system using finite element method, while finite difference method was applied to solve nonlinear Reynolds equation of squeeze film damping. Li *et al*.[9] developed a method for predicting shapes of micro-plates with nonclassical boundary conditions subjected to unsymmetrical electrostatic forces. Jajalli *et al*.[10] investigated dynamic pull-in behaviour of plates actuated by suddenly applied electrostatic force. More recently, Xu and Sun [11] presented nonlinear electromechanical coupled dynamic equations of the micro-plate subjected to electrostatic forces. Talebian *et al*.[12] illustrated a distributed parameter system model for study of electrostatically actuated micro-plates. Effects of temperature, stretching and residual stresses on pull-in voltage and natural frequencies were studied.

Present work studies the effects of various parameters on the static and dynamic solution of electromechanically coupled micro-plate subjected to nonlinear electrostatic excitation. A continuous system model based on a constant bias input voltage is employed to obtain the static deflection and modal solution. A reduced-order model is formulated based on Galerkin approach is employed to obtain the static and dynamic model. Single mode approximation is considered to illustrate the result. The results obtained are shown against the outputs of distributed parameter model. The organization of the paper is as follows: section-2 describes the problem formulation and an approximate solution of distributed parameter model. Section-3 presents, reduced order model using Galerkin method. Section-4 deals with results and discussion.

2. Mathematical Analysis

For dynamic modeling of plates, distributed continuous system models and numerical solution techniques are often employed and the response of the deformed plate can be analyzed very accurately. Most of the works using continuous system models in literature aimed at obtaining analytical predictions of the pull-in position and voltages. Compact models based on analytic equations are effective and offer the advantage of immediate results without complex implementation procedures. On the other hand, numerical models have ability to describe the device by using large number of degrees of freedom so as to represent the real geometry faithfully and investigate effective solution in whole volume. Consider a micro-plate subjected to electrostatic excitation as shown in Fig.1. We assume that (i) both plates are perfect conductors and are separated by a dielectric layer of permittivity ε, (ii) the bottom plate is rigid, and the top one is flexible, (iii) the plate is either clamped, simply-supported or free on the boundary, (iv) a potential difference V exists between the two plates, (v) electric fringing fields are negligible and (vi) the uniform initial gap, d, between the two plates is much smaller than a typical linear dimension of the plate.

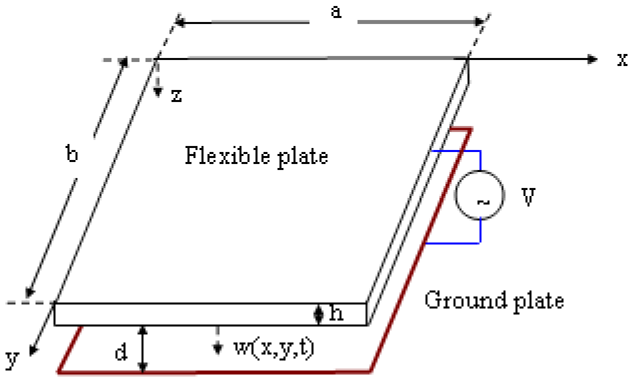

Fig.1 Micro- plate subjected to electro-static force

Under these assumptions the governing equation for the transverse deflection w(x,y,t) of the plate becomes:

$$D\left[\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2 \partial y^2}+\frac{\partial^4 w}{\partial y^4}\right]+\rho h\frac{\partial^2 w}{\partial t^2}=f(x,y,t) \quad (1)$$

where ρ is material density of the plate, t is time, $D=\frac{Eh^3}{12(1-\upsilon^2)}$ is bending stiffness (flexural modulus) of the plate with thickness equal to h, E as elastic modulus of plate material and υ is Poisson ratio. The force per unit area term f(x,y,t) comprises of transverse electrostatic and fluid-film components as well as in-plane normal and shear loads . The electrostatic force is defined according to

$$q(x,y,t)=-\frac{1}{2}V^2\frac{dC}{dw} \quad (2)$$

Here V is the applied voltage and $C=\frac{\varepsilon A}{d-w}$ is capacitance between plates in terms of plate area A, initial gap d and ε is permittivity of free space in air gap. Thus, the electrostatic force per unit area becomes $\frac{\varepsilon V^2}{2(d-w)^2}$. The position w=0 is referred to the undeformed shape of the structure and the ultimate position w=d is provided by dedicated elements (stop dimples) that are able to avoid contact between electrodes.

2.1 Static Analysis

By considering only the electrostatic force, the static governing equation in non-dimensional form can be written as:

$$\left[\frac{\partial^4 W}{\partial \xi^4}+\frac{2}{\psi^2}\frac{\partial^4 W}{\partial \xi^2 \partial \eta^2}+\frac{1}{\psi^4}\frac{\partial^4 W}{\partial \eta^4}\right]=\frac{\varepsilon a^4 V^2}{2Dd^3(1-W)^2}=p(\xi,\eta) \quad (3)$$

Here W=w/d, ξ=x/a, η=y/b and ψ=b/a, are the dimensionless variables considered for convenience. Galerkin's method can be now employed to study the nonlinear response of plate. By considering the evenly distributed electrostatic load on the micro-plate, the dimensionless deflection is considered in multi-mode expansion form [13] as: $W=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}\phi_{mn}$, where A_{mn} is displacement amplitude and ϕ_{mn} is plate displacement function. For rectangular plates with different boundary conditions, these functions are given by Gorman [14]. For example, a plate with four edges under simply-supported boundary conditions, this shape function can be written as ϕ_{mn}=sin mπξ sin nπη and the corresponding expression for transverse deflection becomes:

$$W(\xi,\eta)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}\sin m\pi\xi \sin n\pi\eta \tag{4}$$

The load function p(ξ, η) can also be given as [13]:

$$p(\xi,\eta)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}B_{mn}\sin m\pi\xi \sin n\pi\eta \tag{5}$$

where $$B_{mn}=4\int_0^1\int_0^1 \frac{\varepsilon a^4V^2}{2Dd^3(1-W)^2}\sin m\pi\xi \sin n\pi\eta\, d\xi d\eta$$

$$=\frac{4\varepsilon a^4V^2}{2Dd^3mn\pi^2(1-W)^2}(1-\cos m\pi)(1-\cos n\pi) \tag{6}$$

Substituting Eqs.(4) and (5) in Eq.(3), we get:

$$\pi^4\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(m^2+\frac{n^2}{\psi^2})^2A_{mn}\sin m\pi\xi \sin n\pi\eta = p(\xi,\eta) \tag{7}$$

Substituting p(ξ, η) from Eqs.(5) and (6), we get:

$$A_{mn}=\frac{2\varepsilon a^4V^2}{Dd^3mn\pi^6(1-W)^2(m^2+\frac{n^2}{\psi^2})^2}(1-\cos m\pi)(1-\cos n\pi) \tag{8}$$

Thus, the normalized deflection of the plate is given by substituting A_{mn} in Eq.(4):

$$W(\xi,\eta)=\frac{8\varepsilon a^4V^2}{Dd^3\pi^6(1-W)^2}\sum_{m=1,3,..}^{\infty}\sum_{n=1,3,..}^{\infty}\frac{\sin m\pi\xi \sin n\pi\eta}{mn(m^2+\frac{n^2}{\psi^2})^2} \tag{9}$$

The average deflection of the plate can be obtained as:

$$\overline{W}(\xi,\eta)=\int_0^1\int_0^1 W(\xi,\eta)\,d\xi d\eta \tag{10}$$

2.2 Modal Analysis

When micro- plate is under electrostatic load, its mode shapes and natural frequencies are influenced by applied voltage and initial gap in addition to the geometrical dimensions. Dynamic equation of motion of the electrostatically excited micro- plate is written in terms of dynamic displacement component $w_d(x,y,t)$ as:

$$\left[\frac{\partial^4 w_d}{\partial x^4}+2\frac{\partial^4 w_d}{\partial x^2\partial y^2}+\frac{\partial^4 w_d}{\partial y^4}\right]+\frac{\rho h}{D}\frac{\partial^2 w_d}{\partial t^2}=\frac{1}{D}q_d(x,y,t) \tag{11}$$

Here, $q_d(x,y,t)$ is the dynamic electrostatic flux. The expression for $q_d(x,y,t)$ can be obtained from two term expansion of Eq.(2) by considering voltage V as constant [15]. That is: $q(x,y,t)=-\frac{1}{2}V^2\left(\frac{dC}{dw}+\frac{d^2C}{dw^2}w_d\right)$. The first term has already been accounted as static force, while the second term comes under $q_d(x,y,t)$. Thus,

$$q_d(x,y,t)= -\frac{1}{2}V^2\frac{d^2C}{dw^2}w_d = \frac{\varepsilon V^2 w_d}{(d-\overline{w})^3} \tag{12}$$

Here $\overline{w} = \overline{W}d$, is the average displacement of the plate obtained from static analysis. Using the set of non-dimensional variables: $W_d=w_d/d$, $\xi=x/a$, $\eta=y/b$ and $\psi=b/a$, the resultant dynamic equation reduces to:

$$\frac{\partial^4 W_d}{\partial\xi^4}+\frac{2}{\psi^2}\frac{\partial^4 W_d}{\partial\xi^2\partial\eta^2}+\frac{1}{\psi^4}\frac{\partial^4 W_d}{\partial\eta^4}+\frac{\rho h a^4}{D}\frac{\partial^2 W_d}{\partial t^2}=\frac{\varepsilon a^4 V^2 W_d}{Dd^3(1-\overline{W})^3} \tag{13}$$

In order to solve Eq.(13) using modal superposition, the vertical displacement for free vibration of the plate may be expressed as

$$W_d(\xi,\eta,t)= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}\phi_{mn}(\xi,\eta)e^{j\omega_{mn}t} \tag{14}$$

where $\phi_{mn}(\xi,\eta)$ are the mode shapes corresponding to the ith mode in the x-direction and jth mode in the y-direction with associated natural frequency ω_{mn} and A_{mn} is the unknown modal amplitude. Substituting the expression for W_d in Eq.(13), the resulting equation is simplified as:

$$\frac{\partial^4\phi_{mn}}{\partial\xi^4}+\frac{2}{\psi^2}\frac{\partial^4\phi_{mn}}{\partial\xi^2\partial\eta^2}+\frac{1}{\psi^4}\frac{\partial^4\phi_{mn}}{\partial\eta^4}-\frac{a^4}{D}\left(\rho h\omega_{mn}^2+\frac{\varepsilon V^2}{d^3(1-\overline{W})^3}\right)\phi_{mn}=0 \tag{15}$$

Writing $\frac{a^4}{D}\left(\rho h\omega_{mn}^2+\frac{\varepsilon V^2}{d^3(1-\overline{W})^3}\right)=\beta_{mn}$ and considering the mode shape function as $\phi_{mn}=\sin m\pi\xi \sin n\pi\eta$, Eq.(15) can be revised as the following eigenvalue problem:

$$A_{mn}\left[\pi^4\left(m^2+\frac{n^2}{\psi^2}\right)^2-\beta_{mn}\right]\sin m\pi\xi\,\sin n\pi\eta=0 \tag{16}$$

As $A_{mn}\neq 0$, it gives:

$$\left[\pi^4\left(m^2+\frac{n^2}{\psi^2}\right)^2-\beta_{mn}\right]=0 \tag{16a}$$

Substituting β_{mn} and simplifying,

$$\Omega_{mn}^2=\frac{\omega_{mn}^2}{\overline{\omega}^2}=\frac{a^4\rho h\omega_{mn}^2}{D}=\left[\pi^4\left(m^2+\frac{n^2}{\psi^2}\right)^2-\frac{\varepsilon a^4V^2}{d^3D(1-\overline{W})^3}\right] \tag{17}$$

This gives dimensionless natural frequency parameter Ω_{mn} of the plate. With appropriate initial conditions, the complete dynamic solution can be obtained using Eq.(14).

3. Reduced-order model

There are several approximate solution techniques available in literature to discretize the distributed parameter equation of the plate into finite degrees of freedom such as Galerkin method, Collocation method, Differential Quadrature (DQ) technique etc. In formulating using Galerkin method, the original dynamic equation of motion in continuous form is revised as:

$$(1\text{-}W)^2\left[\frac{\partial^4 W}{\partial \xi^4}+\frac{2}{\psi^2}\frac{\partial^4 W}{\partial \xi^2 \partial \eta^2}+\frac{1}{\psi^4}\frac{\partial^4 W}{\partial \eta^4}\right]+\frac{\rho h a^4}{D}\frac{\partial^2 W}{\partial t^2}=\alpha V^2 \qquad (18)$$

where $\alpha=\dfrac{\varepsilon a^4}{2Dd^3}$. Considering a trial function $W(\xi,\eta,t)=\sum_{m=1}^{M}\sum_{n=1}^{N} q_{mn}(t)\phi_{mn}(\xi,\eta)$ with ϕ_{mn} as mode shape, Galerkin's method is implemented by substituting W in Eq.(18) and then both sides are multiplied by ϕ_{mn} as weighing functions. On double integrating the outcome, from ξ=0 to 1 and η=0 to 1, it results in M×N second-order ordinary differential equations in terms of q_{mn}. For example, with one mode approximation, the following equation is obtained:

$$\int_0^1\!\!\int_0^1\left[\frac{\rho h a^4}{D}\phi_{11}\ddot{q}_{11}+q_{11}\left(\frac{\partial^4 \phi_{11}}{\partial \xi^4}+\frac{2}{\psi^2}\frac{\partial^4 \phi_{11}}{\partial \xi^2 \partial \eta^2}+\frac{1}{\psi^4}\frac{\partial^4 \phi_{11}}{\partial \eta^4}\right)\right]\phi_{11}(1-2\phi_{11}q_{11}+\phi_{11}^2)d\xi d\eta=\alpha V^2\int_0^1\!\!\int_0^1\phi_{11}d\xi d\eta \qquad (19)$$

Here the overdot means derivative with respect to time t. If we can consider ϕ_{11}=sin $\pi\xi$ sin $\pi\eta$ then, the term $\left(\frac{\partial^4 \phi_{11}}{\partial \xi^4}+\frac{2}{\psi^2}\frac{\partial^4 \phi_{11}}{\partial \xi^2 \partial \eta^2}+\frac{1}{\psi^4}\frac{\partial^4 \phi_{11}}{\partial \eta^4}\right)$ becomes: $\pi^4\left(1+\frac{1}{\psi^2}\right)^2\phi_{11}$.

The resulting equation can be simplified as:

$$\left[\frac{\rho h a^4}{D}\ddot{q}_{11}+\pi^4\left(1+\frac{1}{\psi^2}\right)^2 q_{11}\right]\left[\frac{1}{4}-\frac{32}{9\pi^2}q_{11}+\frac{9}{64}q_{11}^2\right]=\frac{4\alpha V^2}{\pi^2} \qquad (20)$$

The static equilibrium equations are obtained by setting the time-derivative to zero in this equation. Thus, the equilibrium equation is

$$\pi^6\left(1+\frac{1}{\psi^2}\right)^2\left(\frac{q_{11}}{4}-\frac{32}{9\pi^2}q_{11}^2+\frac{9}{64}q_{11}^3\right)=4\alpha V^2 \qquad (21)$$

The solution of this equation is obtained algebraically, while the time-domain solution of Eq.(20) can be achieved using either a numerical integration scheme or perturbation method. Such a dynamic solution also facilitates the evaluation of dynamic pull-in voltage of the system.

4. Results and discussions

In order to illustrate the methodology, the following geometric and material properties of micro-plate under electric field are employed [12]: length a=250μm, width b=250 μm, thickness h=3μm, Young's modulus E=169GPa, Poisson's ratio μ=0.06, density ρ=2331 kg/m^3, Initial gap d=1μm, dielectric constant $\varepsilon=8.8541878\times10^{-12}$ F/m. The effects of plate dimensions and initial gap on static pull-in voltage are studied by the two approaches. Fig.2 shows the variation of central static normalized deflection of the plate as a function of applied voltage for different values of aspect ratios.

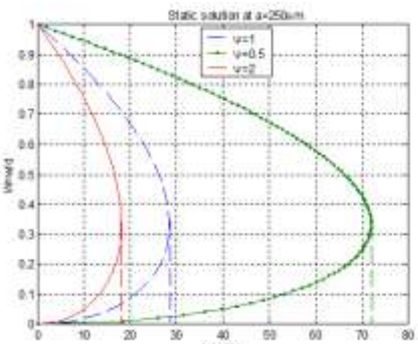

Fig.2 Central deflection of plate versus applied voltage

As seen from the figure, higher the aspect ratio the pull-in voltage reduces. Fig.3 shows variation of pull-in voltage as a function of thickness to gap ratio for a square plate. The variation is nonlinear up to certain value of h/d and later-on it proportionately increases.

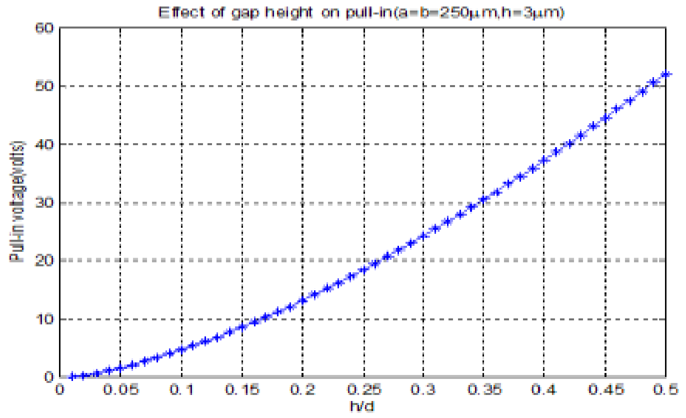

Fig.3 Variation of pull-in voltage with gap height

Fig.4 shows variation of non-dimensional natural frequency at the stable equilibrium points of a square plate as function of applied voltage. It can be seen that by increasing the applied voltage, the natural frequencies of the micro-plate decrease and approaches zero in the vicinity of the pull-in voltage. This frequency reduction from nominal value is based on the fact that there is a reduction in stiffness due to electrostatic force.

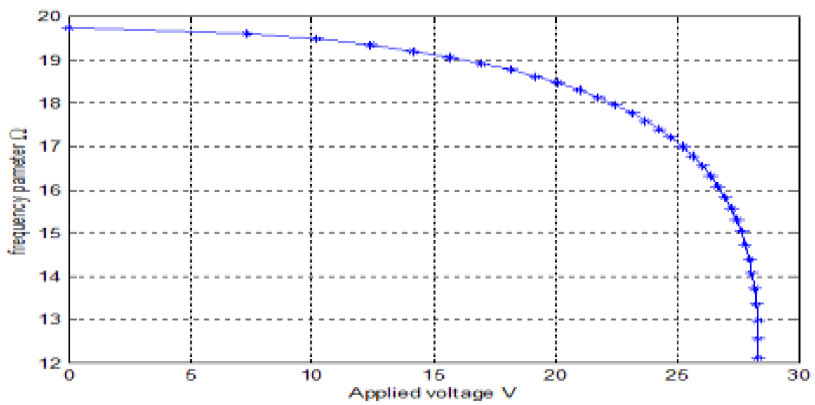

Fig.4 Fundamental natural frequency versus bias voltage

Fig.5 shows the solution of equilibrium equation as obtained from one mode approximate solution. Compared with solution of distribution-parameter model, the one-mode reduced order model predicts the pull-in point at 48% of the gap between the electrode and plate, which is larger than the 33% predicted in former case as reported earlier in literature.

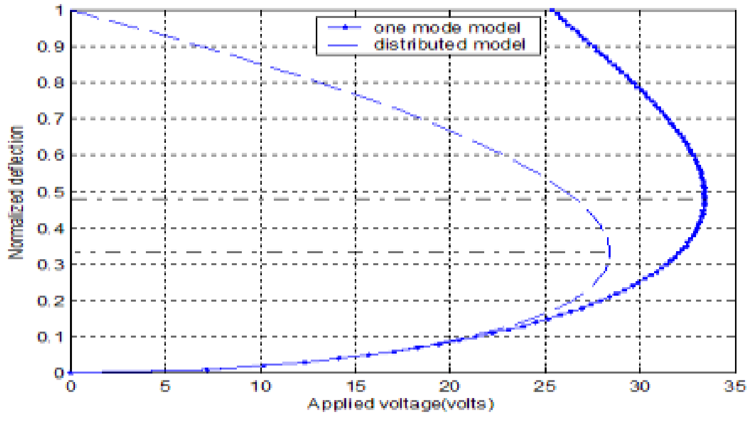

Fig.5 Reduced order solution versus continuous model output

As the number of modes used in the reduced-order modeling increases, the asymptotic convergence can be obtained over the full extent of the lower and upper branches. Fig.6 shows the pull-in surface as a function of center-gap (d/h) ratio of the square plate.

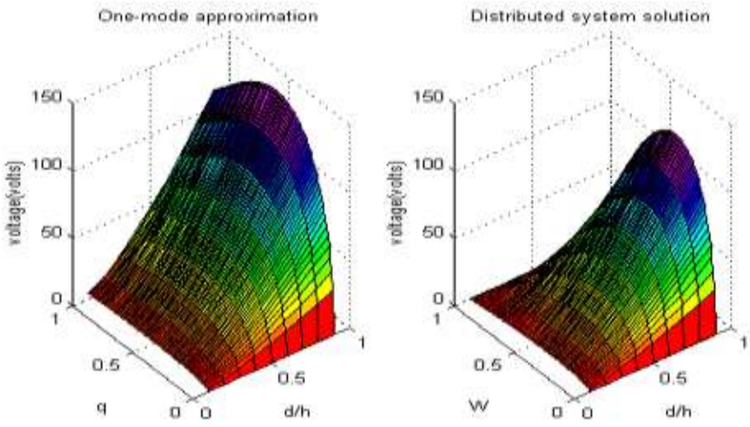

Fig.6 Comparison of pull-in surfaces

Fig.7 shows non-dimensional central deflection history at three different input voltages using reduced-order model.

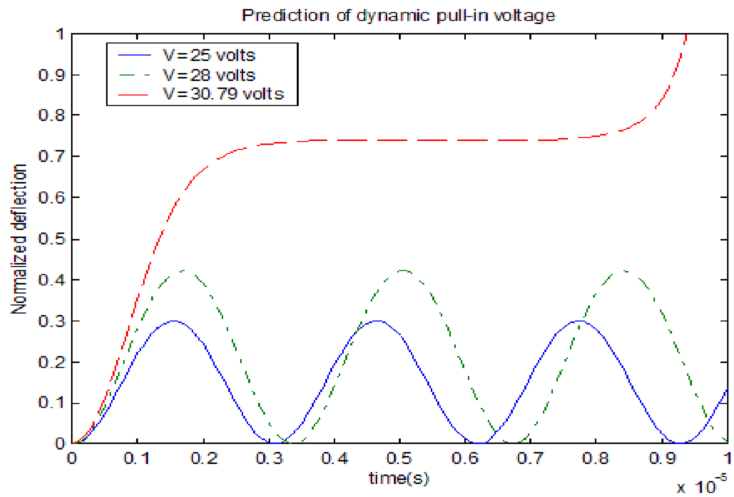

Fig.7 Dynamic solution from reduced order model

As is seen, at a bias voltage 30.79 volts, amplitude shoots-up to very high value indicating the state of dynamic pull-in, which is quite smaller than the static pull-in voltage predicted by reduced-order model earlier.

5. Conclusion

The nonlinear electromechanical coupled dynamics of simply-supported micro-plate subjected to electrostatic force has been presented in this work. The static and dynamic solutions for the closed form equation of motion were arrived analytically. Galerkin's approximate solution approach rendered the equation in reduced form. The static displacements and dynamic natural frequency of simply–supported square plate has been illustrated numerically. It was seen that the static pull-in voltage increases with aspect ratio of plates as well as with initial gap at constant plate thickness. Also, the intrinsic size dependence of material leads to increase in natural frequencies of the micro-plates. A coupled-field finite element analysis has to be employed as a further validation tool.

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