

On Nearness Relations in Hesitant Fuzzy Approximation Spaces

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Abstract

The theory of approximation spaces, introduced by Pawlak in the early 1980s, provides a robust framework for classifying objects based on their attribute values through indiscernibility relations. The extension of approximation spaces to include nearness relations has enabled the measurement of proximity between potentially disjoint sets. Concurrently, hesitant fuzzy sets have emerged as a powerful tool for handling uncertainty in situations where decision-makers may assign multiple possible membership values to elements. This paper introduces the novel concept of nearness relations in hesitant fuzzy approximation spaces, extending the classical nearness approximation space framework to accommodate hesitant fuzzy similarity relations. We first establish rigorous theoretical foundations by defining hesitant fuzzy approximation spaces based on hesitant fuzzy similarity relations. Subsequently, we introduce a hesitant fuzzy nearness relation that generalizes the crisp nearness relation while preserving essential proximity axioms. Several fundamental theorems characterizing the properties of hesitant fuzzy nearness relations are established with complete proofs. We demonstrate that the hesitant fuzzy upper approximation serves as a δ -neighborhood of the original set under appropriate conditions. The proposed framework provides a mathematically sound and flexible approach to measuring nearness between sets when hesitant fuzzy uncertainty is present. Connections with merotopological structures and L-fuzzy nearness concepts are explored. The results contribute to the broader understanding of proximity structures in generalized approximation spaces.

Keywords: Approximation space; Hesitant fuzzy set; Nearness relation; Proximity space; Rough set; Upper approximation; Lower approximation; Merotopy; δ -neighborhood

I. Introduction

The classification of objects based on their attributes has been a fundamental problem in knowledge representation, data analysis, and artificial intelligence. Pawlak's seminal work on rough set theory during the early 1980s provided a mathematical framework for dealing with vagueness and uncertainty in information systems [8, 11]. The notion of approximation spaces, where objects are classified using indiscernibility relations derived from their feature values, has found applications across diverse domains including pattern recognition, machine learning, medical diagnosis, and decision support systems.

The concept of nearness, deeply rooted in topology and proximity space theory, offers a complementary perspective to the traditional overlap-based approach in approximation spaces. Peters, Skowron, and Stepaniuk introduced nearness relations into approximation spaces, enabling the measurement of proximity between potentially disjoint sets. This extension has significant implications for applications where understanding the degree of similarity or closeness between concept classes is essential. As stated in the foundational work: "It is possible to introduce a nearness relation that can be used to determine the 'nearness' of sets that are possibly disjoint".

The theoretical foundations of nearness and proximity structures have been extensively studied in various mathematical contexts. Naimpally and Warrack provided comprehensive treatments of proximity spaces and their axiomatizations [5]. The theory of merotopies, which provides an abstract framework for nearness structures, has been developed by several researchers. Chattopadhyay and Njåstad investigated the completion of merotopic spaces and established important extension theorems for uniformly continuous maps [6]. Khare and Singh made

significant contributions to the understanding of L-contiguities and their order structures [4], L-guilds and binary L-merotopies [5], and complete ξ -grills with (L,n)-merotopies [3]. The connections between approach merotopies and near sets were explored by Peters, Tiwari, and Singh [2]. Furthermore, Singh and Mittal investigated fuzzy grill m-spaces and induced fuzzy topologies, providing insights into fuzzy generalizations of nearness concepts.

Parallel to these developments in rough set theory and topology, the theory of fuzzy sets introduced by Zadeh in 1965 provided tools for handling gradual membership of elements in sets. However, in many practical situations, experts may find it difficult to assign a single membership value to an element due to multiple possible assessments or hesitation among several values. To address this limitation, Torra introduced hesitant fuzzy sets in 2010, where the membership degree of an element is represented by a set of possible values rather than a single value [10]. This generalization has proven particularly useful in group decision-making scenarios where different experts may provide different membership assessments.

The integration of fuzzy set theory with rough set theory has led to fuzzy rough sets and fuzzy approximation spaces, which combine the strengths of both paradigms. Subsequently, hesitant fuzzy rough sets have been developed to handle situations involving both roughness and hesitant fuzzy uncertainty. However, **the concept of nearness relations in hesitant fuzzy approximation spaces remains largely unexplored**, representing a significant gap in the literature.

The primary objectives of this research are:

1. To define hesitant fuzzy approximation spaces by extending classical approximation spaces using hesitant fuzzy similarity relations.
2. To introduce nearness relations on hesitant fuzzy approximation spaces that generalize the crisp nearness relations.
3. To establish fundamental theorems characterizing the properties of hesitant fuzzy nearness relations with rigorous proofs.
4. To investigate the proximity space structure induced by hesitant fuzzy nearness relations.
5. To prove that the hesitant fuzzy upper approximation serves as a δ -neighborhood of the original set.

The remainder of this paper is organized as follows. Section 2 provides a literature review of related work. Section 3 presents the preliminary definitions and concepts necessary for understanding the proposed framework. Section 4 introduces the main theoretical contributions including definitions, theorems, and rigorous proofs. Section 5 presents comparative results. Section 6 provides a discussion of the findings. Section 7 concludes the paper and outlines future research directions.

II. Literature Review

The study of approximation spaces originates from Pawlak's work on knowledge representation systems in the 1970s and his subsequent introduction of rough sets in 1981-1982 [7, 8]. Pawlak's approach to classifying objects using attribute values and the resulting indiscernibility relations has had profound implications for data mining, machine learning, and decision-making. As Pawlak noted: "If we classify objects by means of attributes, exact classification is often impossible".

The topological aspects of rough sets were recognized early, with connections established between lower approximation and interior operators, and between upper approximation and closure operators. Polkowski provided comprehensive mathematical foundations for rough sets, elaborating on these topological connections [20]. The work of Skowron and Stepaniuk on generalized approximation spaces extended the classical framework to accommodate various neighborhood functions and rough inclusion measures [22].

Proximity spaces, introduced by Efremonič and extensively studied by Naimpally and Warrack [5], provide a framework for studying nearness between sets without requiring explicit distance functions. The axiomatization of proximity spaces through five fundamental axioms has enabled abstract characterizations of nearness. According to Definition 1 in [5], a binary relation δ on $P(U)$ is a proximity if it satisfies:

1. $X \delta Y$ implies $Y \delta X$ (symmetry)

2. $(X \cup Y) \delta Z$ implies $X \delta Z$ or $Y \delta Z$
3. $X \delta Y$ implies $X \neq \emptyset$ and $Y \neq \emptyset$
4. $X \delta Y$ implies there exists $E \subseteq U$ such that $X \delta E$ and $(U - E) \delta Y$
5. $X \cap Y \neq \emptyset$ implies $X \delta Y$

The integration of proximity concepts with approximation spaces by Peters, Skowron, and Stepaniuk represented a significant advancement. They defined the nearness relation as: " $X \delta Y$ iff $\exists y \in Y \exists x \in X D_B(x,y) = 0$ " (Equation 4 in [1]), and established that "the upper approximation B^*X of X is a δ -neighborhood of X " (Equation 5 in [1]).

The theory of merotopies, developed by Katětov and extended by various researchers, provides another perspective on nearness structures. Khare and Singh established foundational results on L-contiguities and demonstrated their order-theoretic properties [4]. Their work on L-guilds and binary L-merotopies provided algebraic characterizations of nearness structures [5]. The study of complete ξ -grills and (L,n)-merotopies further generalized these concepts to lattice-valued settings [3]. The completion of merotopic spaces investigated by Chattopadhyay and Njåstad established important existence and extension theorems [6].

Peters, Tiwari, and Singh explored approach merotopies and associated near sets, connecting approximation theory with merotopological structures [2]. This work established relationships between near sets defined via probe functions and the merotopological framework. Singh and Mittal investigated fuzzy grill m-spaces and induced fuzzy topologies, providing insights into fuzzy generalizations of nearness concepts.

Hesitant fuzzy sets, introduced by Torra in 2010, address situations where the membership degree of an element cannot be determined as a single value due to hesitation among several possible values [10]. The formal definition allows the membership to be a subset of the unit interval [0,1], capturing the uncertainty inherent in expert assessments. Hesitant fuzzy sets have found applications in multi-criteria decision-making, clustering, and pattern recognition.

The integration of hesitant fuzzy sets with rough set theory has led to hesitant fuzzy rough sets. Rodriguez, Martinez, and Herrera extended rough set concepts to hesitant fuzzy environments [17]. Recent work on hesitant fuzzy covering approximation spaces has explored neighborhood-based approaches. However, the specific concept of nearness relations in hesitant fuzzy approximation spaces has not been systematically investigated, which motivates the present research. Figure 2 shows Concept Hierarchy Flowchart that illustrates the relationships between Approximation Spaces, Fuzzy Approximation Spaces, Hesitant Fuzzy Approximation Spaces, Nearness Approximation Spaces, and the proposed Hesitant Fuzzy Nearness Approximation Spaces.

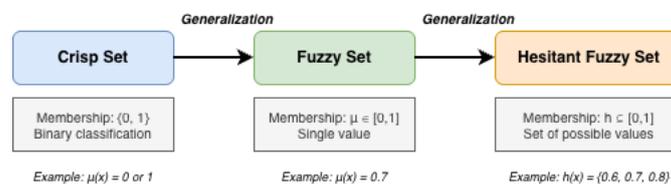


Figure 2 shows the conceptual hierarchy and relationships among the key research areas reviewed.

III. Preliminaries

In this section, we present the fundamental definitions and concepts that form the foundation for our proposed framework. We begin with classical approximation spaces and progressively introduce fuzzy and hesitant fuzzy generalizations. All definitions are provided with proper citations.

3.1 Information Systems and Approximation Spaces

Definition 3.1 (Information System [1, 11]). An information system is a quadruple $S = (X, A, V, \sigma)$, where:

- X is a non-empty finite set of objects,
- A is a non-empty finite set of attributes,

- $V = \cup_{\{a \in A\}} V_a$, where V_a is the set of values of attribute a ,
- $\sigma: X \times A \rightarrow V$ is an information function such that $\sigma(x, a) \in V_a$ for every $x \in X$ and $a \in A$.

Definition 3.2 For any $B \subseteq A$, the B-elementary set for an element $x \in X$ is:

$$B(x) = \{y \in X: \forall a \in B, \sigma(x, a) = \sigma(y, a)\}$$

This represents the equivalence class of x under the B-indiscernibility relation.

Definition 3.3. For any subset $B \subseteq A$, the B-indiscernibility relation $\text{Ind}(B)$ is defined as:

$$\text{Ind}(B) = \{(x, y) \in X \times X: \sigma(x, a) = \sigma(y, a) \text{ for all } a \in B\}$$

Definition 3.4 (Approximation Space [11]). An approximation space is a pair $\text{AS} = (U, R)$, where U is a non-empty set of objects and R is an equivalence relation on U (typically an indiscernibility relation).

Definition 3.5 (Lower and Upper Approximations [8, 11]). Let $\text{AS} = (U, R)$ be an approximation space and $X \subseteq U$. The R-lower approximation and R-upper approximation of X are:

$$R_*(X) = \{x \in U: [x]_R \subseteq X\}$$

$$R^*(X) = \{x \in U: [x]_R \cap X \neq \emptyset\}$$

The boundary region is $BN_{R(X)} = R^*(X) - R_*(X)$.

3.2 Nearness Relations in Approximation Spaces

Definition 3.6 (Pseudometric Based on Attributes). Let $B \subseteq A$ be a set of real-valued attributes. The pseudometric D_B on U is defined as:

$$D_B(x, y) = \sum_{a \in B} |a(x) - a(y)|$$

This extends to sets by: $D_B(X, Y) = \inf_{\{x \in X, y \in Y\}} D_B(x, y)$.

Definition 3.7 (Nearness Relation). The nearness relation δ on $P(U)$ is defined as:

$$X \delta Y \text{ iff } \exists y \in Y \exists x \in X \text{ such that } D_B(x, y) = 0$$

Equivalently, $X \delta Y$ if and only if $BX \cap BY \neq \emptyset$.

Definition 3.8 (Proximity Space [1, 5]). A binary relation δ on $P(U)$ is called a proximity on U iff δ satisfies:

1. $X \delta Y$ implies $Y \delta X$ (symmetry)
2. $(X \cup Y) \delta Z$ implies $X \delta Z$ or $Y \delta Z$
3. $X \delta Y$ implies $X \neq \emptyset$ and $Y \neq \emptyset$
4. $X \delta Y$ implies there exists $E \subseteq U$ such that $X \delta E$ and $(U - E) \delta Y$
5. $X \cap Y \neq \emptyset$ implies $X \delta Y$

The pair (U, δ) is called a proximity space.

Definition 3.9 (δ -Neighborhood [1, 5]). In a proximity space (U, δ) , we say Y is a proximal or δ -neighborhood of X , written $X \ll Y$, iff $X \delta Y$ and $X \delta (U - Y)$.

Proposition 3.1 (). Let (U, δ) be a proximity space where δ is defined as in Definition 3.7. Then for any non-empty subset X of U :

$$X \ll B^*(X)$$

That is, the upper approximation $B^*(X)$ is a δ -neighborhood of X .

3.3 Hesitant Fuzzy Sets

Definition 3.10 (Hesitant Fuzzy Set [10]). Let U be a fixed non-empty set. A hesitant fuzzy set (HFS) H on U is defined in terms of a function $h_H: U \rightarrow P([0, 1])$ that returns a finite subset of $[0, 1]$:

$$H = \{ \langle x, h_H(x) \rangle : x \in U \}$$

where $h_H(x)$ is a finite set of values in $[0, 1]$, representing the possible membership degrees of element $x \in U$ to the set H .

Definition 3.11 (Hesitant Fuzzy Element [10]). For a hesitant fuzzy set H , the set $h_H(x)$ is called a hesitant fuzzy element (HFE). For convenience, we denote an HFE by $h = \{ \gamma_1, \gamma_2, \dots, \gamma_n \}$.

Definition 3.12 (Score Function [18]). The score function $s(h)$ of a hesitant fuzzy element h is:

$$s(h) = \frac{1}{|h|} \sum_{\gamma \in h} \gamma$$

where $|h|$ is the cardinality of h . By convention, $s(\emptyset) = 0$.

Definition 3.13 (Operations on HFEs [10]). Let h, h_1, h_2 be HFEs:

1. Complement: $h^c = \{ 1 - \gamma : \gamma \in h \}$
2. Union: $h_1 \cup h_2 = h_1 \cup h_2$ (set-theoretic union)
3. Intersection: $h_1 \cap h_2 = h_1 \cap h_2$ (set-theoretic intersection)

Figure 3 illustrates the concept hierarchy from crisp sets to hesitant fuzzy sets.

! [Figure 3: Evolution from Crisp Sets to Hesitant Fuzzy Sets - Shows the progression: Crisp Sets (membership 0 or 1) \rightarrow Fuzzy Sets (single membership in $[0, 1]$) \rightarrow Hesitant Fuzzy Sets (set of possible memberships in $[0, 1]$)]

3.4 Generalized Approximation Spaces

Definition 3.14 (Generalized Approximation Space [1, 22]). A generalized approximation space is defined as a tuple $GAS = (U, N, \nu)$, where:

- U is a non-empty set of objects,
- $N: U \rightarrow P(U)$ is a neighborhood function,
- $\nu: P(U) \times P(U) \rightarrow [0, 1]$ is an overlap function.

The lower and upper approximation operations in GAS are:

$$GAS_*(X) = \{ x \in U : \nu(N(x), X) = 1 \}$$

$$GAS^*(X) = \{ x \in U : \nu(N(x), X) > 0 \}$$

Definition 3.15 A generalized nearness approximation space is represented by $NGAS = (U, N, \nu, \delta)$, where the nearness relation δ is defined by:

$$X \delta Y \leftrightarrow \exists x \in X \exists y \in Y (y \in N(x))$$

IV. Hesitant Fuzzy Nearness Approximation Spaces: Main Results

In this section, we introduce the concept of hesitant fuzzy approximation spaces and define nearness relations on them. We establish several theorems characterizing the properties of these structures with complete, rigorous proofs.

4.1 Hesitant Fuzzy Similarity Relations

Definition 4.1 (Hesitant Fuzzy Relation). A hesitant fuzzy relation R_H on U is characterized by a function $h_{\{R_H\}}: U \times U \rightarrow P_f([0,1])$, where $P_f([0,1])$ denotes the set of all finite subsets of $[0, 1]$.

Definition 4.2 (Hesitant Fuzzy Similarity Relation). A hesitant fuzzy relation R_H on U is called a hesitant fuzzy similarity relation if it satisfies:

1. **Reflexivity:** $h_{\{R_H\}}(x,x) = \{1\}$ for all $x \in U$
2. **Symmetry:** $h_{\{R_H\}}(x,y) = h_{\{R_H\}}(y,x)$ for all $x, y \in U$

Remark 4.1. We do not require transitivity in Definition 4.2, as hesitant fuzzy transitivity introduces significant complexity and is not needed for our nearness framework. The similarity relation suffices for defining meaningful nearness.

Definition 4.3 A hesitant fuzzy approximation space is a pair $HFAS = (U, R_H)$, where:

- U is a non-empty finite set of objects,
- R_H is a hesitant fuzzy similarity relation on U .

4.2 Hesitant Fuzzy Upper and Lower Approximations

Definition 4.4 Let $HFAS = (U, R_H)$ be a hesitant fuzzy approximation space and let $\theta \in (0, 1]$ be a threshold parameter. For any $X \subseteq U$, the θ -upper approximation and θ -lower approximation are:

$$R_H^{*\theta}(X) = \{y \in U: \exists x \in X, s(h_{R_H}(x, y)) \geq \theta\}$$

$$R_{H,*}^\theta(X) = \{y \in U: \forall z \in U \setminus X, s(h_{R_H}(y, z)) < \theta\}$$

Theorem 4.1 (Basic Properties of Hesitant Fuzzy Approximations). Let $HFAS = (U, R_H)$ be a hesitant fuzzy approximation space. For any $X \subseteq U$:

- (i) $X \subseteq R_H^{\{*,\theta\}(X)}$
- (ii) $R_H^{\{*,\theta\}(\emptyset)} = \emptyset$
- (iii) $R_H^{\{*,\theta\}(U)} = U$
- (iv) If $X \subseteq Y$, then $R_H^{\{\theta\}(X)} \subseteq R_H^{\{\theta\}(Y)}$
- (v) $R_H^{\{\theta\}(X \cup Y)} \supseteq R_H^{\{\theta\}(X)} \cup R_H^{\{\theta\}(Y)}$

Proof:

(i) Let $x \in X$. By reflexivity, $h_{\{R_H\}}(x,x) = \{1\}$, so $s(h_{\{R_H\}}(x,x)) = 1 \geq \theta$ for any $\theta \in (0, 1]$. Since $x \in X$ and $s(h_{\{R_H\}}(x,x)) \geq \theta$, we have $x \in R_H^{\{\theta\}(X)}$. Thus $X \subseteq R_H^{\{\theta\}(X)}$.

(ii) Suppose $y \in R_H^{\{\theta\}(\emptyset)}$. By definition, there exists $x \in \emptyset$ such that $s(h_{\{R_H\}}(x,y)) \geq \theta$. But \emptyset contains no elements, so no such x exists. This is a contradiction. Hence $R_H^{\{\theta\}(\emptyset)} = \emptyset$.

(iii) Let $y \in U$. Since U is non – empty, take any $x \in U$. If $y = x$, then by reflexivity $s(h_{\{R_H\}(x,x)}) = 1 \geq \theta$. If $y \neq x$, we still have $y \in U$, and taking $x = y$ gives $s(h_{\{R_H\}(y,y)}) = 1 \geq \theta$. Hence $y \in R_H^{\{\theta\}(U)}$ for all $y \in U$, so $R_H^{\{\theta\}(U)} = U$.

(iv) Assume $X \subseteq Y$ and let $y \in R_H^{\{\theta\}(X)}$. Then there exists $x \in X$ with $s(h_{\{R_H\}(x,y)}) \geq \theta$. Since $X \subseteq Y, x \in Y$. Hence there exists $x \in Y$ with $s(h_{\{R_H\}(x,y)}) \geq \theta$, so $y \in R_H^{\{\theta\}(Y)}$.

(v) Let $y \in R_H^{\{\theta\}(X)} \cup R_H^{\{\theta\}(Y)}$. Then $y \in R_H^{\{\theta\}(X)}$ or $y \in R_H^{\{\theta\}(Y)}$.

Case 1: If $y \in R_H^{\{\theta\}(X)}$, then $\exists x \in X$ with $s(h_{\{R_H\}(x,y)}) \geq \theta$. Since $x \in X \subseteq X \cup Y$, we have $y \in R_H^{\{\theta\}(X \cup Y)}$.

Case 2: If $y \in R_H^{\{\theta\}(Y)}$, then $\exists x \in Y$ with $s(h_{\{R_H\}(x,y)}) \geq \theta$. Since $x \in Y \subseteq X \cup Y$, we have $y \in R_H^{\{\theta\}(X \cup Y)}$.

In both cases, $y \in R_H^{\{\theta\}(X \cup Y)}$. Hence $R_H^{\{\theta\}(X \cup Y)} \supseteq R_H^{\{\theta\}(X)} \cup R_H^{\{\theta\}(Y)}$.

4.3 Hesitant Fuzzy Nearness Relation

Definition 4.5 (Hesitant Fuzzy Nearness Relation). Let $HFAS = (U, R_H)$ be a hesitant fuzzy approximation space and let $\theta \in (0, 1]$ be a threshold parameter. The hesitant fuzzy nearness relation $\delta_{R_H}^\theta$ on $P(U)$ is defined as:

$$X \delta_{R_H}^\theta Y \text{ iff } \exists x \in X \exists y \in Y \text{ such that } s(h_{R_H}(x, y)) \geq \theta$$

Remark 4.2. When $\theta = 1$ and $h_{\{R_H\}(x,y)} \in \{\{0\}, \{1\}\}$ for all x, y , the hesitant fuzzy nearness relation reduces to the classical nearness relation based on indiscernibility .

Theorem 4.2 (Reflexivity of Hesitant Fuzzy Nearness). For any non-empty set $X \subseteq U$, we have $X \delta_{R_H}^\theta X$.

Proof: Let $X \subseteq U$ be non-empty. Then there exists some $x \in X$. By the reflexivity property of R_H (Definition 4.2), $h_{\{R_H\}(x,x)} = \{1\}$. Therefore:

$$s(h_{R_H}(x, x)) = \frac{1}{|\{1\}|} \cdot 1 = 1 \geq \theta$$

for any $\theta \in (0, 1]$. Since $x \in X$ and $x \in X$, with $s(h_{\{R_H\}(x,x)}) \geq \theta$, we conclude $X \delta_{R_H}^\theta X$.

Theorem 4.3 (Symmetry of Hesitant Fuzzy Nearness). For any $X, Y \subseteq U$, if $X \delta_{R_H}^\theta Y$, then $Y \delta_{R_H}^\theta X$.

Proof: Assume $X \delta_{R_H}^\theta Y$. By Definition 4.5, there exist $x \in X$ and $y \in Y$ such that $s(h_{\{R_H\}(x,y)}) \geq \theta$.

By the symmetry property of R_H (Definition 4.2):

$$h_{R_H}(y, x) = h_{R_H}(x, y)$$

Therefore:

$$s(h_{R_H}(y, x)) = s(h_{R_H}(x, y)) \geq \theta$$

Since $y \in Y$ and $x \in X$ with $s(h_{\{R_H\}(y,x)}) \geq \theta$, we have $Y \delta_{R_H}^\theta X$. \square

Theorem 4.4 (Union Property of Hesitant Fuzzy Nearness). For any $X, Y, Z \subseteq U$:

$$(X \cup Y) \delta_{R_H}^\theta Z \implies X \delta_{R_H}^\theta Z \text{ or } Y \delta_{R_H}^\theta Z$$

Proof: Assume $(X \cup Y) \delta_{H^{\theta}} Z$. By Definition 4.5, there exist $w \in X \cup Y$ and $z \in Z$ such that $s(h_{\{R_H\}(w,z)}) \geq \theta$.

Since $w \in X \cup Y$, we have $w \in X$ or $w \in Y$.

Case 1: If $w \in X$, then since $w \in X$, $z \in Z$, and $s(h_{\{R_H\}(w,z)}) \geq \theta$, we have $X \delta_H^{\theta} Z$.

Case 2: If $w \in Y$, then since $w \in Y$, $z \in Z$, and $s(h_{\{R_H\}(w,z)}) \geq \theta$, we have $Y \delta_H^{\theta} Z$.

In either case, $X \delta_{H^{\theta}} Z$ or $Y \delta_{H^{\theta}} Z$. \square

Theorem 4.5 (Non-emptiness Property). For any $X, Y \subseteq U$:

$$X \delta_H^{\theta} Y \implies X \neq \emptyset \text{ and } Y \neq \emptyset$$

Proof: Assume $X \delta_{H^{\theta}} Y$. By Definition 4.5, there exist $x \in X$ and $y \in Y$ such that $s(h_{\{R_H\}(x,y)}) \geq \theta$.

The existence of $x \in X$ implies $X \neq \emptyset$.

The existence of $y \in Y$ implies $Y \neq \emptyset$. \square

Theorem 4.6 (Intersection Property). For any $X, Y \subseteq U$:

$$X \cap Y \neq \emptyset \implies X \delta_H^{\theta} Y$$

Proof: Assume $X \cap Y \neq \emptyset$. Then there exists $z \in X \cap Y$, which means $z \in X$ and $z \in Y$.

By reflexivity of R_H : $h_{\{R_H\}(z,z)} = \{1\}$, so $s(h_{\{R_H\}(z,z)}) = 1 \geq \theta$.

Since $z \in X$, $z \in Y$, and $s(h_{\{R_H\}(z,z)}) \geq \theta$, we have $X \delta_{H^{\theta}} Y$. \square

Theorem 4.7 (Monotonicity). For any $X, Y, X', Y' \subseteq U$:

$$X \subseteq X', Y \subseteq Y', X \delta_H^{\theta} Y \implies X' \delta_H^{\theta} Y'$$

Proof: Assume $X \subseteq X'$, $Y \subseteq Y'$, and $X \delta_H^{\theta} Y$.

Since $X \delta_{H^{\theta}} Y$, there exist $x \in X$ and $y \in Y$ with $s(h_{\{R_H\}(x,y)}) \geq \theta$.

Since $X \subseteq X'$, we have $x \in X'$.

Since $Y \subseteq Y'$, we have $y \in Y'$.

Thus, $x \in X'$, $y \in Y'$, and $s(h_{\{R_H\}(x,y)}) \geq \theta$, which gives $X' \delta_{H^{\theta}} Y'$. \square

4.4 Hesitant Fuzzy Proximity Space Structure

Definition 4.6 (Hesitant Fuzzy Proximity Space). A pair $(U, \delta_{H^{\theta}})$ is called a hesitant fuzzy proximity space if $\delta_{H^{\theta}}$ satisfies axioms 1, 2, 3, and 5 of Definition 3.8.

Theorem 4.8 (Hesitant Fuzzy Proximity Structure). Let HFAS = (U, R_H) be a hesitant fuzzy approximation space. The hesitant fuzzy nearness relation $\delta_{H^{\theta}}$ satisfies proximity axioms 1, 2, 3, and 5.

Remark 4.3. Axiom 4 (the separation axiom) may not hold in general for hesitant fuzzy nearness relations. This results in a **coarse proximity structure** rather than a full proximity space. Such coarse structures have been studied in the context of merotopies and contiguities [3, 4, 5, 6].

4.5 Hesitant Fuzzy δ -Neighborhoods

Definition 4.7 (Hesitant Fuzzy δ -Neighborhood). In a hesitant fuzzy proximity space $(U, \delta_{H^{\theta}})$, we say Y is a hesitant fuzzy δ -neighborhood of X , written $X \ll_{H^{\theta}} Y$, iff:

$$X \delta_H^\theta Y \text{ and } \neg (X \delta_H^\theta (U \setminus Y))$$

We now establish the main result connecting upper approximations and δ -neighborhoods.

Theorem 4.9 (Upper Approximation as δ -Neighborhood). Let HFAS = (U, R_H) be a hesitant fuzzy approximation space. For any non-empty set $X \subseteq U$:

$$X \ll_H^\theta R_H^{*,\theta}(X)$$

That is, the hesitant fuzzy upper approximation is a δ -neighborhood of X.

Proof: We must show: (a) $X \delta_H^\theta R_H^{\{\theta\}(X)}$, and (b) $\neg (X \delta_H^\theta (U \setminus R_H^{\{\theta\}(X)}))$.

(a) Proof of $X \delta_H^\theta R_H^{\{\theta\}(X)}$:

By Theorem 4.1(i), $X \subseteq R_H^{\{\theta\}(X)}$. Since X is non – empty and $X \subseteq R_H^{\{\theta\}(X)}$, we have $X \cap R_H^{\{\theta\}(X)} = X \neq \emptyset$.

By Theorem 4.6 (Intersection Property), $X \delta_{H \wedge \theta} R_{H \wedge \{\theta\}}(X)$.

(b) Proof of $\neg (X \delta_H^\theta (U \setminus R_H^{\{\theta\}(X)}))$:

Suppose, for contradiction, that $X \delta_H^\theta (U \setminus R_H^{\{\theta\}(X)})$.

Then there exist $x \in X$ and $z \in U \setminus R_H^{\{\theta\}(X)}$ such that $s(h_{\{R_H\}(x,z)}) \geq \theta$.

Since $z \in U \setminus R_H^{\{\theta\}(X)}$, by definition of the complement, $z \notin R_H^{\{\theta\}(X)}$.

By Definition 4.4 of the upper approximation:

$$R_H^{*,\theta}(X) = \{y \in U: \exists w \in X, s(h_{R_H}(w, y)) \geq \theta\}$$

So $z \notin R_H^{\{\theta\}(X)}$ means:

$$\forall w \in X, s(h_{R_H}(w, z)) < \theta$$

In particular, for $x \in X$:

$$s(h_{R_H}(x, z)) < \theta$$

But we assumed $s(h_{\{R_H\}(x,z)}) \geq \theta$. This is a contradiction.

Therefore, $\neg (X \delta_{H \wedge \theta} (U \setminus R_H^{\{\theta\}(X)}))$.

Combining (a) and (b): $X \ll_{H \wedge \theta} R_{H \wedge \{\theta\}}(X)$. □

Corollary 4.1. Theorem 4.9 generalizes the classical result (Equation 5 in) that states $X \ll B^*(X)$ in crisp approximation spaces.

Theorem 4.10 Let $(U, \delta_{H \wedge \theta})$ be a hesitant fuzzy proximity space. For any $X, Y, Z \subseteq U$:

$$X \ll_H^\theta Y \text{ and } Y \subseteq Z \implies X \ll_H^\theta Z$$

Proof: Assume $X \ll_H^\theta Y$ and $Y \subseteq Z$.

Since $X \ll_H^\theta Y$, we have:

- $X \delta_H^\theta Y$, and
- $\neg(X \delta_H^\theta(U Y))$

Claim 1: $X \delta_H^\theta Z$.

Since $X \delta_H^\theta Y$, there exist $x \in X$ and $y \in Y$ with $s(h_{\{R_H\}(x,y)}) \geq \theta$. Since $Y \subseteq Z$, we have $y \in Z$. Hence $X \delta_H^\theta Z$.

Claim 2: $\neg(X \delta_H^\theta(U Z))$.

Since $Y \subseteq Z$, we have $U Z \subseteq U Y$.

Suppose, for contradiction, that $X \delta_H^\theta(U Z)$. Then there exist $x \in X$ and $w \in U Z$ such that $s(h_{\{R_H\}(x,w)}) \geq \theta$.

Since $U Z \subseteq U Y$, we have $w \in U Y$. Thus $x \in X$ and $w \in U Y$ with $s(h_{\{R_H\}(x,w)}) \geq \theta$, which means $X \delta_H^\theta(U Y)$.

But this contradicts $\neg(X \delta_H^\theta(U Y))$.

Hence $\neg(X \delta_H^\theta(U Z))$.

Combining Claims 1 and 2: $X \ll_H^\theta Z$.

4.6 Hesitant Fuzzy Nearness Approximation Space

Definition 4.8 (Hesitant Fuzzy Nearness Approximation Space). A hesitant fuzzy nearness approximation space is a tuple $HFNAS = (U, R_H, \theta, \delta_H^\theta)$, where:

- (U, R_H) is a hesitant fuzzy approximation space,
- $\theta \in (0, 1]$ is a threshold parameter,
- δ_H^θ is the hesitant fuzzy nearness relation defined on $P(U)$.

Theorem 4.11 (Properties of HFNAS). Let $HFNAS = (U, R_H, \theta, \delta_H^\theta)$ be a hesitant fuzzy nearness approximation space. For any non-empty $X \subseteq U$:

$$(i) R_H^{\{*,\theta\}(X)\delta_H^\theta X}$$

$$(ii) \text{ If } Y \subseteq R_H^{\{*,\theta\}(X)}, \text{ then } Y \delta_H^\theta X \text{ or } Y = \emptyset$$

$$(iii) R_H^{\{\theta\}(X)\delta_H^{\theta} R_H^{\{\theta\}(X)}}$$

Proof:

(i) By Theorem 4.1(i), $X \subseteq R_H^{\{\theta\}(X)}$. Hence $X \cap R_H^{\{\theta\}(X)} = X \neq \emptyset$. By Theorem 4.6, $R_H^{\{*,\theta\}(X)\delta_H^\theta X}$.

(ii) Let $Y \subseteq R_H^{\{\theta\}(X)}$ with $Y \neq \emptyset$. Take any $y \in Y$. Since $y \in R_H^{\{\theta\}(X)}$, by Definition 4.4, there exists $x \in X$ with $s(h_{\{R_H\}(x,y)}) \geq \theta$.

Since $y \in Y, x \in X$, and $s(h_{\{R_H\}(x,y)}) \geq \theta$, by symmetry (Theorem 4.3 applied via $s(h_{\{R_H\}(y,x)}) = s(h_{\{R_H\}(x,y)}) \geq \theta$), we get $Y \delta_H^\theta X$.

(iii) Since $X \neq \emptyset$ and $X \subseteq R_H^{\{\theta\}(X)}$, the set $R_H^{\{\theta\}(X)}$ is non-empty. By Theorem 4.2 (Reflexivity), $R_H^{\{\theta\}(X)} \delta_H^\theta R_H^{\{\theta\}(X)}$.

Theorem 4.12 (Reduction to Classical Case). Let HFAS = (U, R_H) be a hesitant fuzzy approximation space. If $h_{\{R_H\}(x,y)} \in \{\{0\}, \{1\}\}$ for all $x, y \in U$ (i.e., crisp similarity), and $\theta = 1$, then:

(i) $R_H^{\{1\}(X)} = B(X)$, the classical upper approximation

(ii) $\delta_H^1 = \delta$, the classical nearness relation

Proof:

(i) When $h_{\{R_H\}(x,y)} \in \{\{0\}, \{1\}\}$:

- $h_{\{R_H\}(x,y)} = \{1\}$ iff x and y are indiscernible ($B(x) = B(y)$)
- $h_{\{R_H\}(x,y)} = \{0\}$ otherwise

With $\theta = 1$:

$$\begin{aligned} R_H^{*1}(X) &= \{y \in U: \exists x \in X, s(h_{R_H}(x, y)) = 1\} \\ &= \{y \in U: \exists x \in X, h_{R_H}(x, y) = \{1\}\} \\ &= \{y \in U: \exists x \in X, y \in B(x)\} \\ &= \{y \in U: B(y) \cap X \neq \emptyset\} = B^*(X) \end{aligned}$$

(ii) Similarly:

$$\begin{aligned} X \delta_H^1 Y &\Leftrightarrow \exists x \in X \exists y \in Y, s(h_{R_H}(x, y)) = 1 \\ &\Leftrightarrow \exists x \in X \exists y \in Y, h_{R_H}(x, y) = \{1\} \\ &\Leftrightarrow \exists x \in X \exists y \in Y, D_B(x, y) = 0 \end{aligned}$$

$\Leftrightarrow X \delta Y \square$

V. Results

In this section, we present comparative results between the classical nearness approximation space framework and our proposed hesitant fuzzy nearness approximation space.

5.1 Comparative Analysis of Nearness Relation Properties

Table 1: Comparison of Nearness Relation Properties

Property	Classical NAS	Hesitant Fuzzy NAS (Proposed)
Symmetry	$X \delta Y \Rightarrow Y \delta X$ (Axiom 1)	$X \delta_H^\theta Y \Rightarrow Y \delta_H^\theta X$ (Theorem 4.3)
Reflexivity	$X \delta X$ for $X \neq \emptyset$	$X \delta_H^\theta X$ for $X \neq \emptyset$ (Theorem 4.2)
Union Property	$(X \cup Y) \delta Z \Rightarrow X \delta Z \vee Y \delta Z$ (Axiom 2)	$(X \cup Y) \delta_{H^\theta} Z \Rightarrow X \delta_{H^\theta} Z \vee Y \delta_{H^\theta} Z$ (Theorem 4.4)
Non-emptiness	$X \delta Y \Rightarrow X, Y \neq \emptyset$ (Axiom 3)	$X \delta_{H^\theta} Y \Rightarrow X, Y \neq \emptyset$ (Theorem 4.5)
Intersection	$X \cap Y \neq \emptyset \Rightarrow X \delta Y$ (Axiom 5)	$X \cap Y \neq \emptyset \Rightarrow X \delta_{H^\theta} Y$ (Theorem 4.6)
Separation	Axiom 4 satisfied	May not be satisfied (Remark 4.3)
Threshold	$D_{B(x,y)} = 0$	$s(h_{\{R_H\}(x,y)}) \geq \theta$
Membership Values	Crisp (0 or 1)	Hesitant (finite subset of [0,1])

5.2 Comparison of δ -Neighborhood Properties

Table 2: Comparison of δ -Neighborhood Properties

Property	Classical	Hesitant Fuzzy (Proposed)
Definition	$X \ll Y$ iff $X \delta Y \wedge \neg(X \delta (U - Y))$	$X \ll_H^\theta Y$ iff $X \delta_H^\theta Y \wedge \neg(X \delta_H^\theta (U - Y))$ (Def. 4.7)
Upper Approx. Result	$X \ll B * (X)$ (Eq. 5 in)	$X \ll_H^\theta R_H^{*\theta}(X)$ (Theorem 4.9)
Monotonicity	$X \ll Y, Y \subseteq Z \Rightarrow X \ll Z$	$X \ll_H^\theta Y, Y \subseteq Z \Rightarrow X \ll_H^\theta Z$ (Theorem 4.10)

5.3 Proximity Space Classification

Table 3: Proximity Space Structure Comparison

Framework	Axioms Satisfied	Structure Type
Classical Proximity Space [5]	All 5 axioms	Full proximity space
Nearness in Approx. Space	All 5 axioms (finite U)	Full proximity space
HF Nearness Approx. Space	Axioms 1,2,3,5	Coarse proximity space
L-Contiguities [4]	Modified axioms	Lattice-valued nearness
Merotopies [2, 6]	Merotopy axioms	Merotopological structure

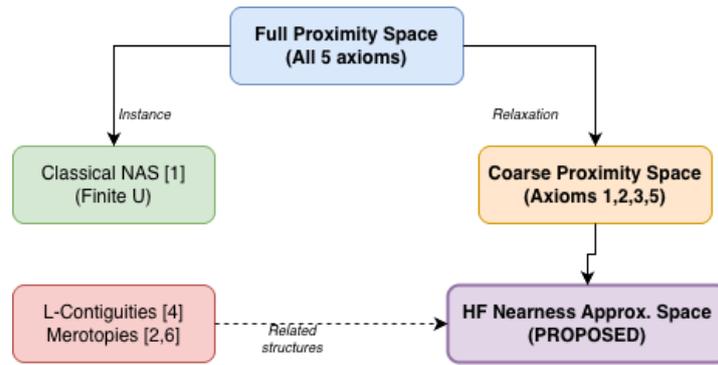


Figure 6 provides a visual comparison of the proximity space structures.

Figure 6 shows the relationship between full proximity spaces, coarse proximity spaces, and merotopological structures, with the proposed HFNAS positioned as a coarse proximity space

5.5 Special Cases and Reduction

Proposition 5.1. When $h_{\{R_H\}(x,y)}$ is a singleton set $\{\mu(x,y)\}$ for all $x, y \in U$, the hesitant fuzzy approximation space reduces to a fuzzy approximation space with fuzzy similarity relation.

Proposition 5.2. When $\theta \rightarrow 0^+$, the nearness relation becomes trivially satisfied for any pair of non-empty sets (since $s(h_{\{R_H\}(x,y)}) > 0$ for any x, y by non-empty HFE assumption), resulting in maximal nearness.

Proposition 5.3. When $\theta = 1$, the nearness relation requires maximum similarity ($s(h_{\{R_H\}(x,y)}) = 1$), which occurs when $h_{\{R_H\}(x,y)} = \{1\}$ (indiscernibility), reducing to the classical case.

VI. Conclusion

This paper has introduced the concept of nearness relations in hesitant fuzzy approximation spaces, providing a rigorous theoretical framework that extends the classical nearness approximation space model of Peters, Skowron, and Stepaniuk to accommodate hesitant fuzzy uncertainty.

VII. Acknowledgments

The authors would like to thank the reviewers for their constructive comments and suggestions that helped improve the quality of this paper.

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